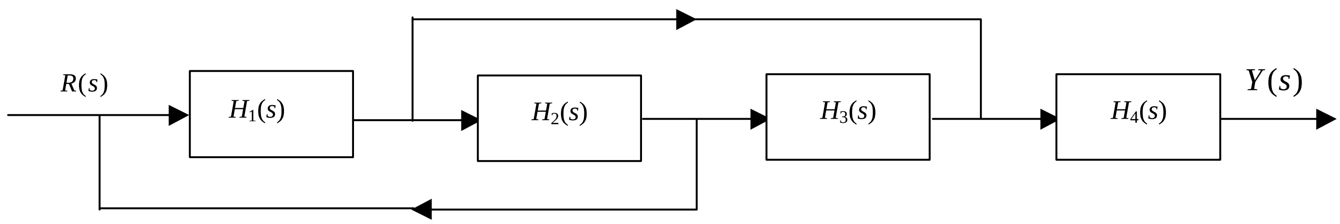
Block Diagram Manipulation [Section 3.2.]

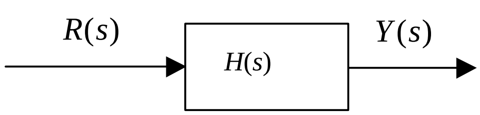
We often represent control systems using block diagrams. A block diagram consists of blocks that represent transfer functions of the different variables of interest.

If a block diagram has many blocks, not all of which are in cascade, then it is useful to have rules for rearranging the diagram such that you end up with only one block.

For example, we would want to transform the following diagram



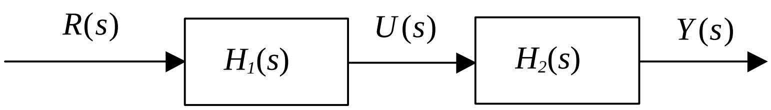

into




How do we get $H(s)$ from $H_1(s),\ H_2(s),\ H_3(s),\ H_4(s)$?

Manipulating and Reducing Block Diagrams [Section 3.2.1]

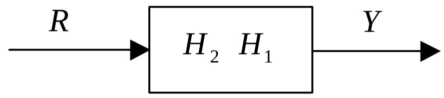
Since each transfer function represents a linear system, their product is commutative, i.e., for the diagram below



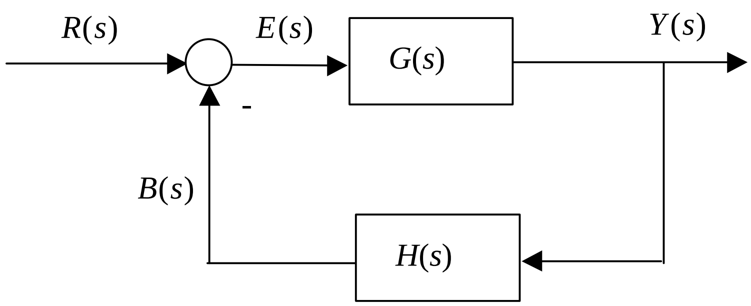

$Y(s) = H_2(s)\ U(s)$ and $U(s) = H_1(s)\ R(s)$
leads to

$$Y(s) = H_2(s)\, H_1(s)\, R(s)$$

so that the above block diagram can be redrawn as




Now, let’s consider a simple feedback loop:




If we write equations for the above diagram we get

$$E(s) = R(s) - B(s) \qquad (1)$$
$$B(s) = H(s) \cdot Y(s) \qquad (2)$$
$$Y(s) = G(s) \cdot E(s) \qquad (3)$$

Substitute (2) into (1) to get

$$E(s) = R(s) - H(s)\,Y(s) \qquad (4)$$

$+$

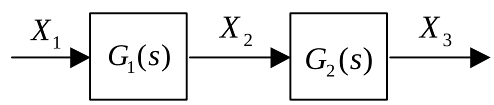
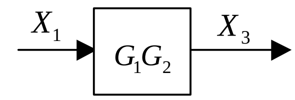
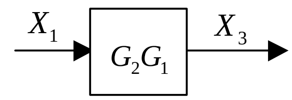
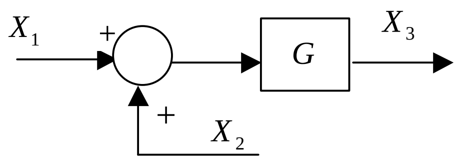
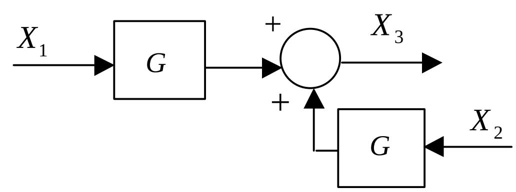
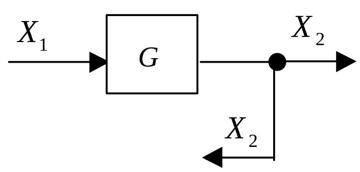
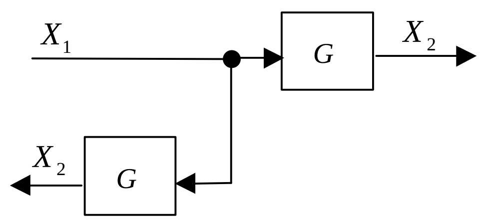
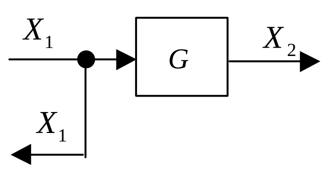
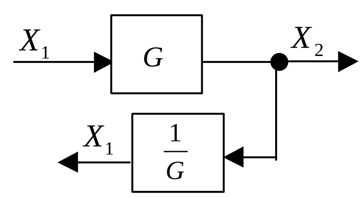
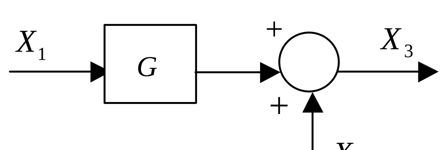
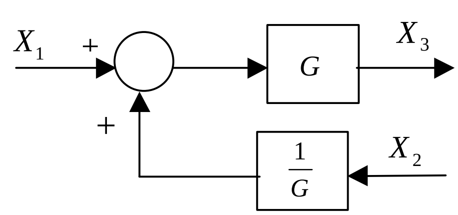
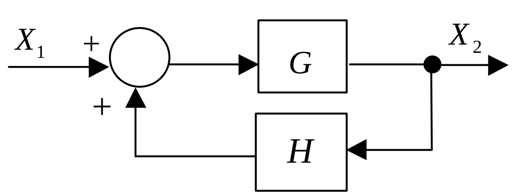
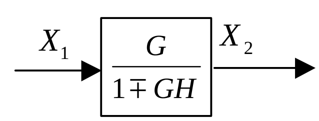
Substitute (4) into (3) to get

$$Y(s) = G(s)\ \left(R(s) - H(s)\ Y(s)\right)$$
i.e., $Y(s)\ \left(1 + G(s)\ H(s)\right) = G(s)\ R(s)$
i.e., $\dfrac{Y(s)}{R(s)} = \dfrac{G(s)}{1 + G(s)\ H(s)}$

The transfer function $\dfrac{G}{1+GH}$ is called the closed-loop transfer function. From the above equation, we can see that the feedback loop can be redrawn as

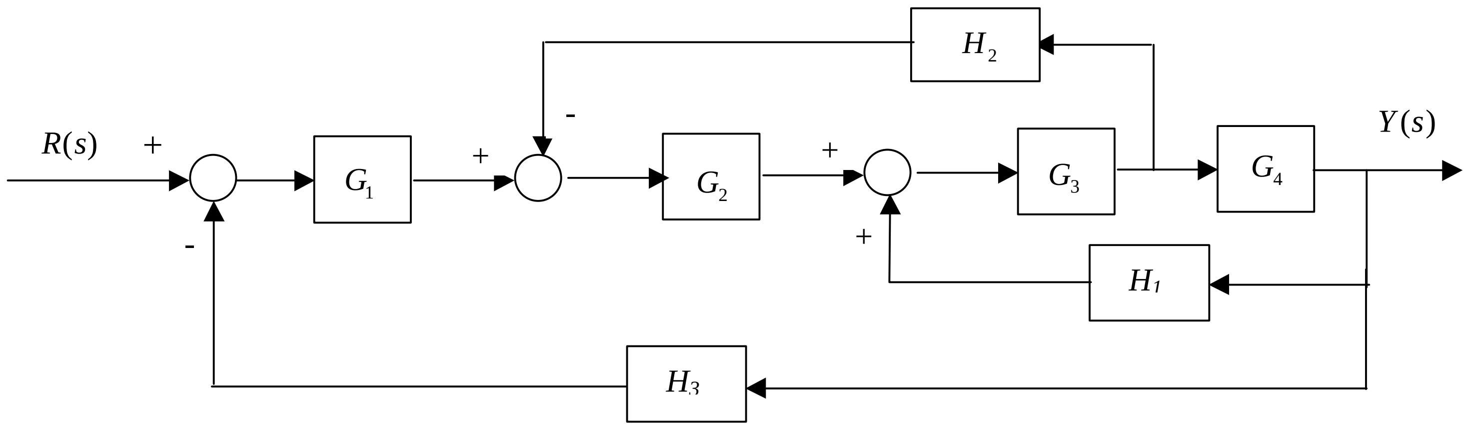

We have just shown two cases (cascade and feedback) of block diagram reduction. These and other transformations are given in Table 1.

**Table 1**. Block Diagram Transformations [Taken from Dorf & Bishop Textbook]

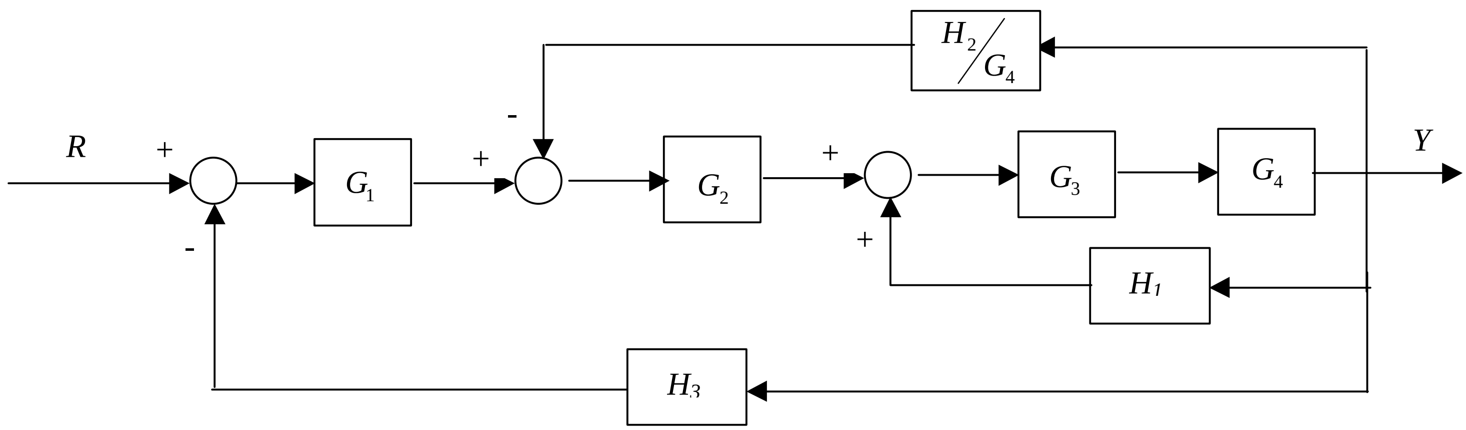
| Transformation | Original Diagram | Equivalent Diagram |
|---|---|---|
| 1. Combining blocks in cascade |  |  or  |
| 2. Moving a summing point behind a block |  |  |
| 3. Moving a pickoff point ahead of a block |  |  |
| 4. Moving a pickoff point behind a block |  |  |
| 5. Moving a summing point ahead of a block |  |  |
| 6. Eliminating a feedback loop |  |  |

Example [Using the equivalence transformations of Table 1]

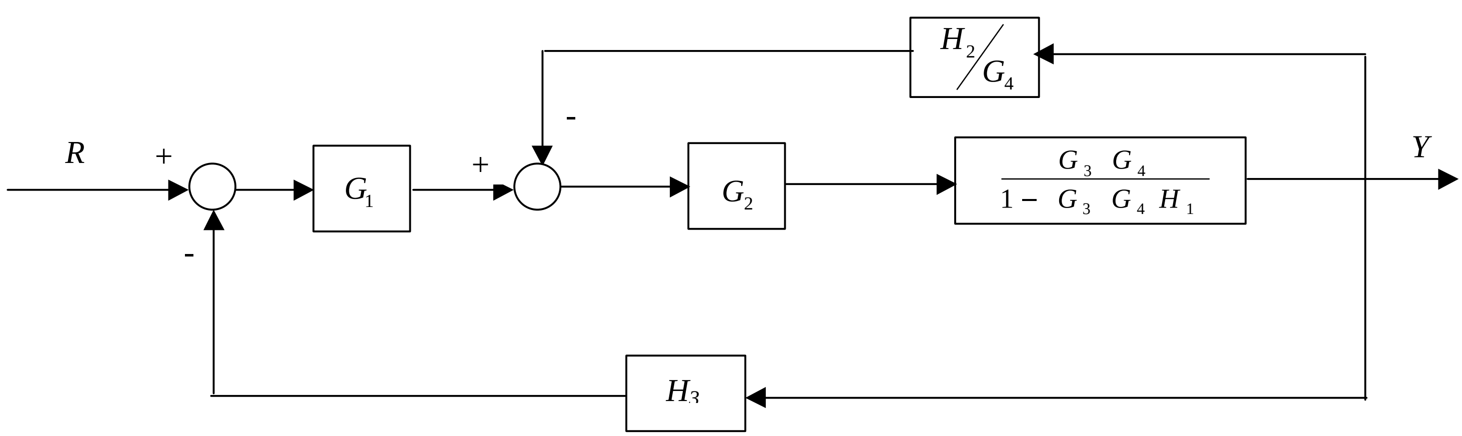
Consider the following feedback control system:




First, let's move $H_2$ behind block $G_4$ so that we can isolate the $G_3 - G_4 - H_1$ feedback loop. Use Rule 4 of Table 1 to get



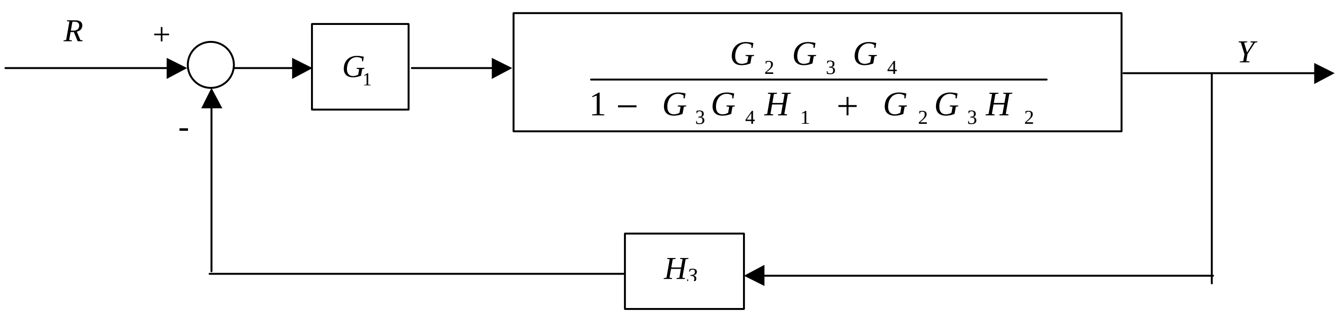

Now, we can eliminate the $G_3 - G_4 - H_1$ loop by using Rule 6 of Table 1:



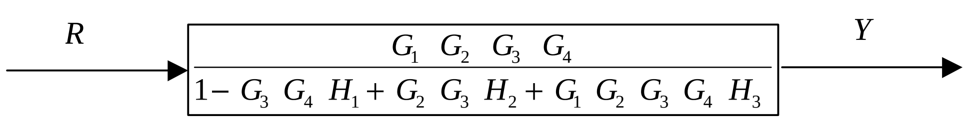
Now, eliminate the inner loop using Rule 6 of Table 1 again:

$$\frac{\dfrac{G_2 G_3 G_4}{1-G_3 G_4 H_1}}{1+\underbrace{\dfrac{G_2 G_3 G_4}{1-G_3 G_4 H_1}}_{G}\underbrace{\dfrac{H_2}{G_4}}_{H}} = \frac{G_2 G_3 G_4}{1-G_3 G_4 H_1 + G_2 G_3 H_2}$$

Therefore, the equivalent diagram is




Finally, we can eliminate the last feedback loop to get



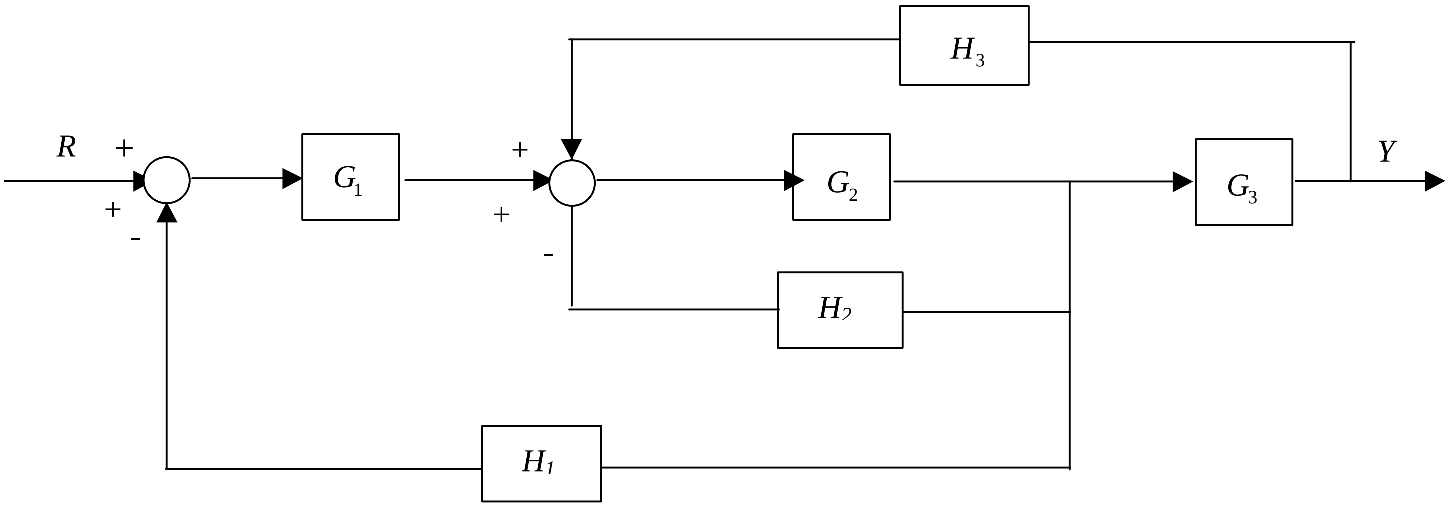

Using block diagrams, it becomes easy to see where new blocks can be added to an existing system to alter system performance.

Note: Signal-flow graphs are an alternate representation of control systems

- Advantage: no need to do iterative and tedious block diagram manipulations
- See Section 3.2.2 of textbook

Alternate Method for Getting Transfer Functions of Multiloop Systems (without having to do block diagram manipulation)

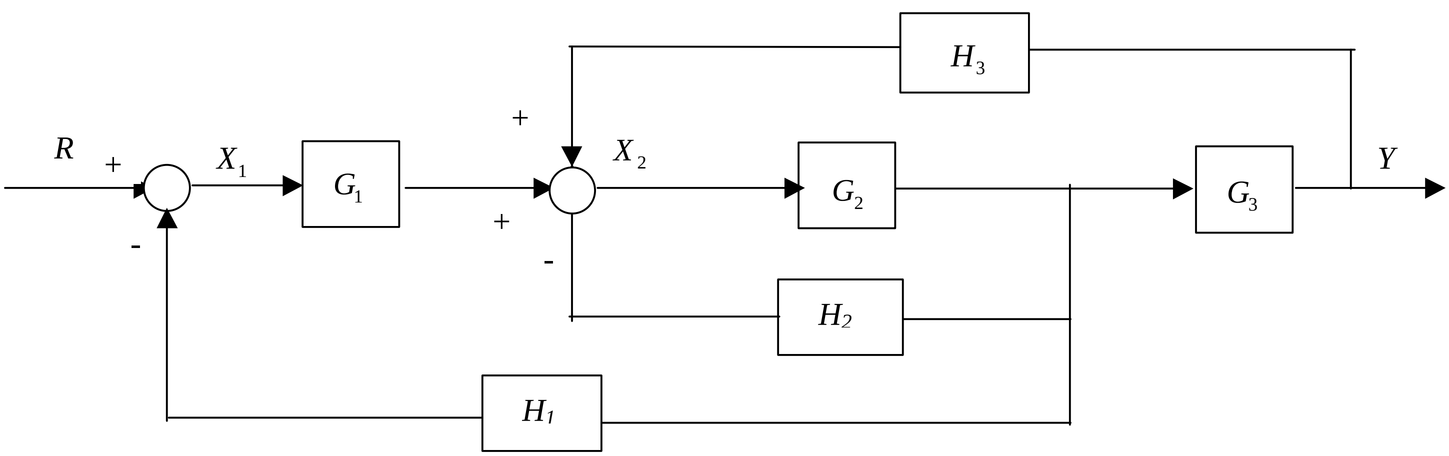
Suppose we want the transfer function from R to Y of the following multi-component system:



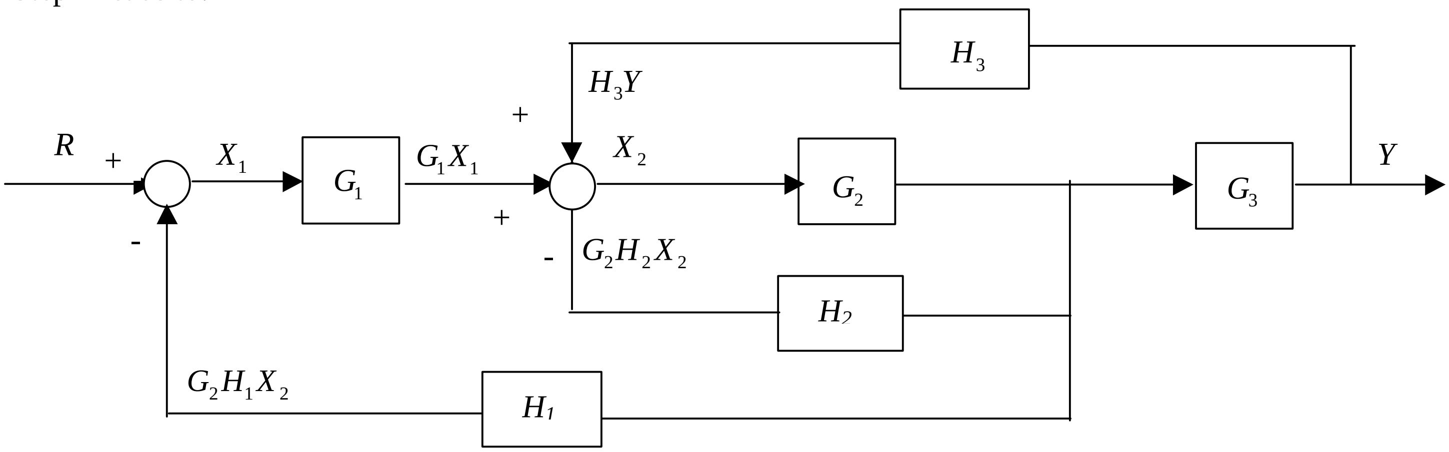

Do the following:

1) Label the outputs of the summing junctions, say $X_i$
2) Label the inputs to the summing junctions in terms of $X_i$ and output Y
3) Write equations at the summing junctions and at the output
4) Eliminate the $X_i$'s

So, for the above example, step 1 leads to:



Step 2 leads to:




Step 3 gives:

$$\begin{bmatrix} 1 & G_2 H_1 & 0 \\ -G_1 & 1+G_2 H_2 & -H_3 \\ 0 & -G_2 G_3 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ Y \end{bmatrix} = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}$$

Then, using Cramer's Rule we can do step 4:

$$Y = \frac{\begin{vmatrix} 1 & G_2 H_1 & R \\ -G_1 & 1+G_2 H_2 & 0 \\ 0 & -G_2 G_3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & G_2 H_1 & 0 \\ -G_1 & 1+G_2 H_2 & -H_3 \\ 0 & -G_2 G_3 & 1 \end{vmatrix}}$$

$$\frac{Y}{R} = \frac{\begin{vmatrix} 1 & G_2 H_1 & 1 \\ -G_1 & 1+G_2 H_2 & 0 \\ 0 & -G_2 G_3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & G_2 H_1 & 0 \\ -G_1 & 1+G_2 H_2 & -H_3 \\ 0 & -G_2 G_3 & 1 \end{vmatrix}}$$

## Cramer's Rule

If $Ax = B$ is a system of $n$ linear equations in $n$ unknowns such that $\det A \neq 0,$ then

$$x_1 = \frac{\det(A_1)}{\det A}\;,\; x_2 = \frac{\det(A_2)}{\det A}\;,\; ----\;,\; x_n = \frac{\det(A_n)}{\det A}$$

Where $A_j$ is the matrix obtained by replacing the entries in the $j^{th}$ column of $A$ by the entries in the matrix

$$B = \begin{bmatrix} b_1 \\ b_2 \\ ' \\ ' \\ ' \\ b_n \end{bmatrix}$$