Control Engineering Cheat Sheet

Innhold

Engelsk	Norsk	Beskrivelse
Process variable	Prosess variabel	Variabelen som blir regulert. (eks. temperatur)
Process value	Prosess verdi	Den nåværende verdien til prosess variabelen.
Set point	Settpunkt	Ønsket verdi for prosess variabelen.
Actuator	Aktuator	En innretning som gjør om en form for energi til en annen. (eks. ventil, sylinder)
Plant/System	System	Matematisk modell som representerer det fysiske systemet.
Saturation	Metning	Fysisk realiserbare systemer vil alltid ha en grense på hvor mye pådrag du kan gi. Saturation er en "hard limit" for pådraget.
Wind-up		Wind-up er et problem som kan skje på grunn av integral delen i PID regulatorer. På grunn av metning kan integral leddet hope seg opp og det vil bli stor overshoot.
Dead time	Dødtid	Dødtid er forsinkelsen fra set point er endret til prosess variabelen forandres.
Bandwidth	Båndbredde	Båndbredde frekvensen er definert som den frekvensen closed-loop magnituden er -3 dB. Båndbredden er bæremengden av frekvenser til systemet som kan holde på nyttig informasjon.
Crossover frequency	Kryssfrekvens	Kryssfrekvensen er der amplituden krysser 0 dB i ett bode plot.
Natural response/Transient response	Naturlig respons	Er systemets respons til startverdibetingelsene.
Natural frequency	Egenfrekvens	Den frekvensen der det vil oppstå resonans.
Forced response	Tvungen respons	Er systemets respons til de eksterne påvirkningene uten tanke på startverdibetingelsene.
Steady State	Stabil tilstand	Den tilstanden systemet er i når prosess verdien endrer seg lite og har gått mot en bestemt verdi.
Steady State Error		Forskjellen mellom setpoint og steady state verdien.
Rise time		Tiden det tar fra 10 % av final value til 90 %.
Settling time		Tiden det tar før responsen forblir innfor ±2 %.
Peak time		Tiden det tar før den først peaken blir nådd.
Asymptote	Asymptote	Geometrisk forklart er en asymptote en tangentlinje til en kurve i punktet uendelig.
Gain	Forsterkning	
Gain margin	Forsterkningsmargin	Antall dB mellom 0 dB og den frekvens hvor fasen er -180°.
Stability margins	Stabilitetsmarginer	Gain og phase margin.
Phase margin	Fasemargin	Antall grader mellom -180° og fasen ved kryssfrekvensen.
Resonance	Resonans	Resonans er det at et system oscillerer med større amplitude ved gitte frekvenser.
Decade	Dekade	En dekade er en faktor på 10 differanse mellom to tall.
Octave	Oktav	Dobling eller halvering av frekvens.
Unity feedback		Feedback med 1 i gain.

Linearization

$$f(x) = f(x_0) + \left.\frac{df(x)}{dx}\right|_{x=x_0} (x - x_0)$$

Ex.

$$\frac{d^2\theta(t)}{dt^2} + \sin\theta(t) = 0; \qquad f(0) = \sin\theta,\ \ \theta_0 = 0$$

We only need to linearize the sine part of the equation because $\frac{d^2\theta(t)}{dt^2}$ is already linear;

$$\sin\theta = \sin\theta_0 + \cos\theta_0 \cdot (\theta - \theta_0)$$

Systems

Linear System

$$s(\alpha_1 u_1 + \alpha_2 u_2) = a_1 \cdot s(u_1) + \alpha_2 \cdot s(u_2)$$

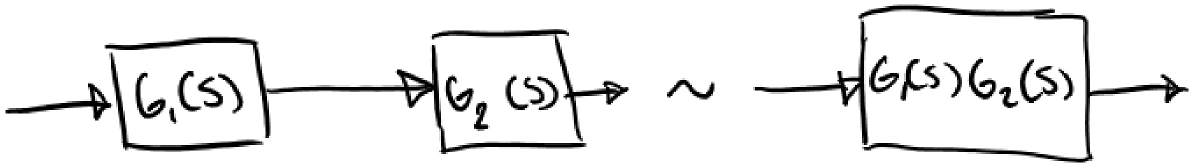
Time-varying System

$$y(t+\tau) = s(u(t+\tau)$$

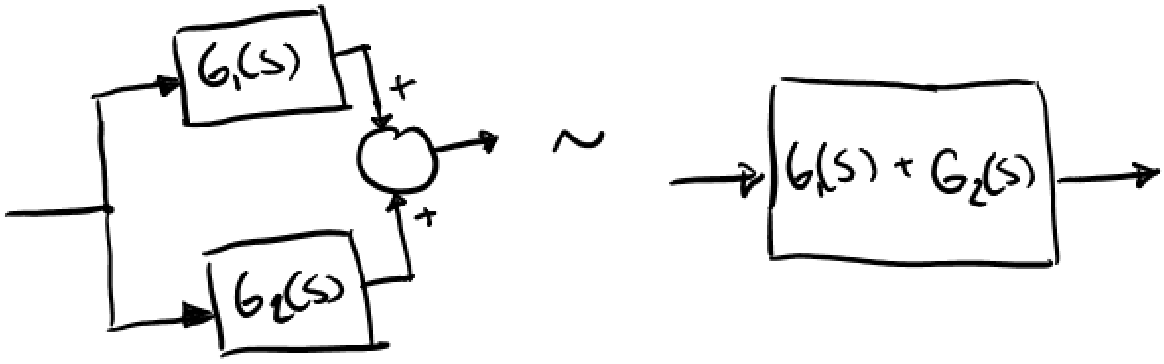
Block Diagrams

Block Diagram Manipulation

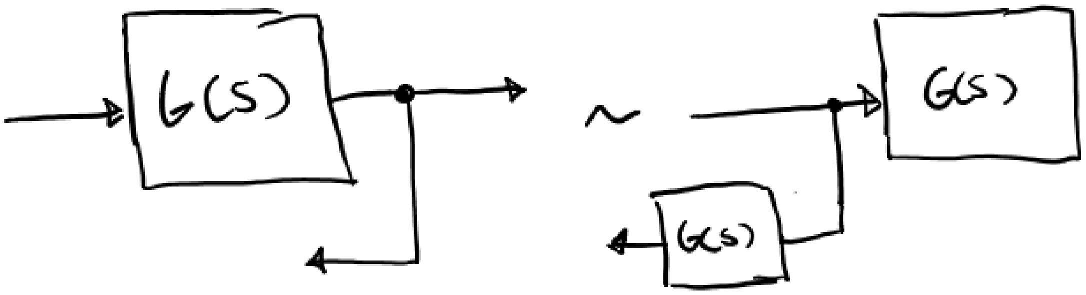
Rule 1: Combining Cascade Blocks



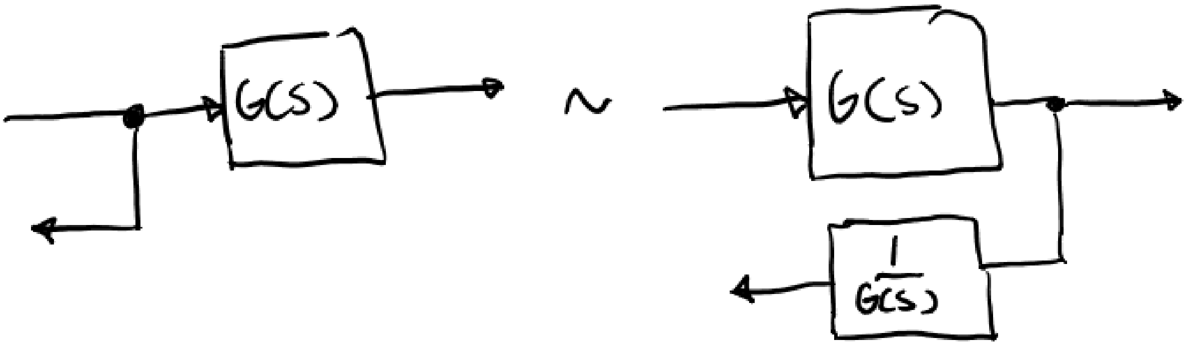
Rule 2: Eliminating a Forward Loop



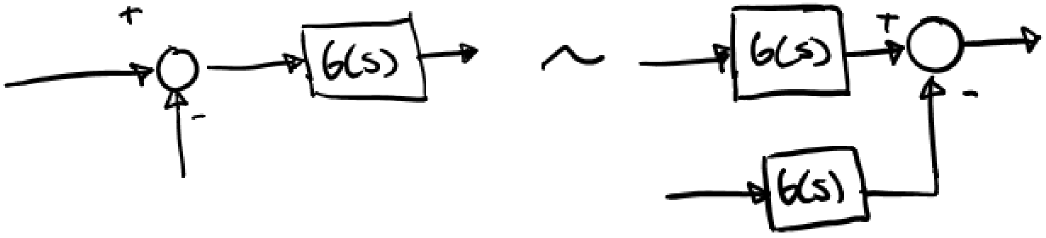
Rule 3: Moving a Branch-point



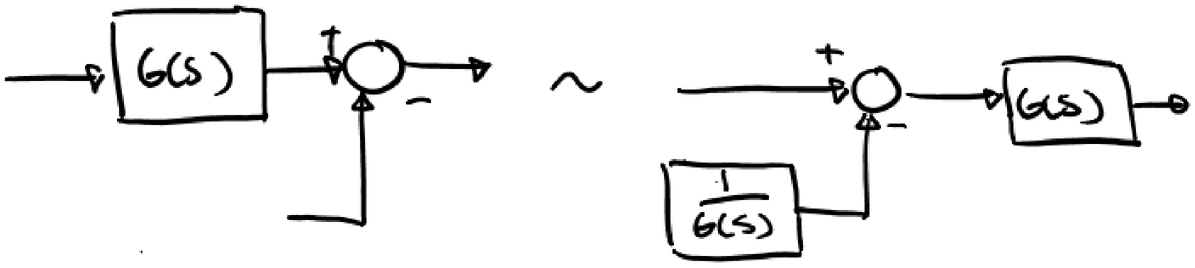
and



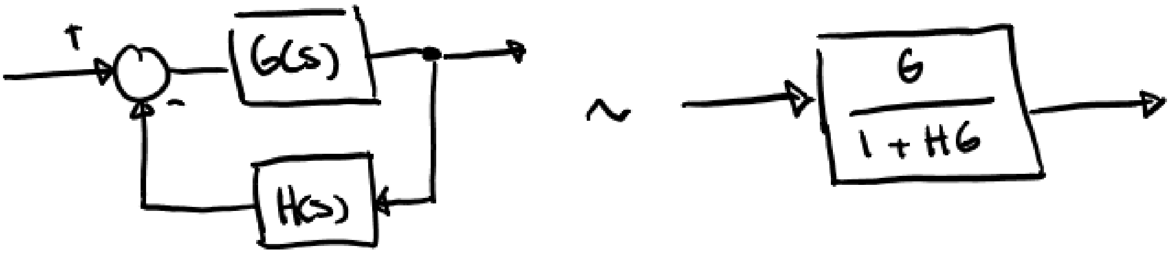
Rule 4: Moving a Summing-points



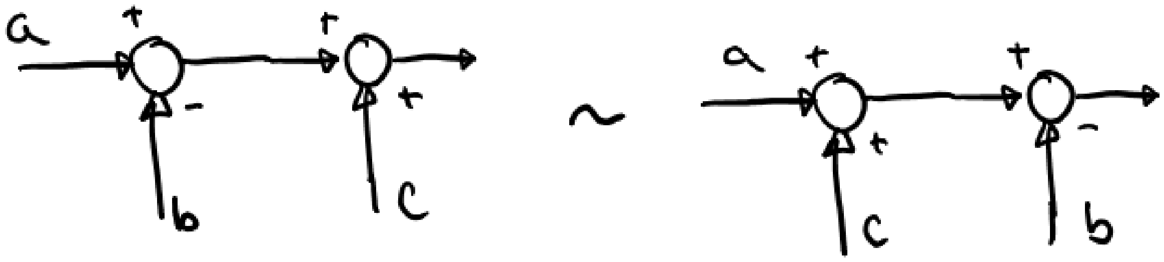
and



Rule 5: Eliminating Feedback Loops

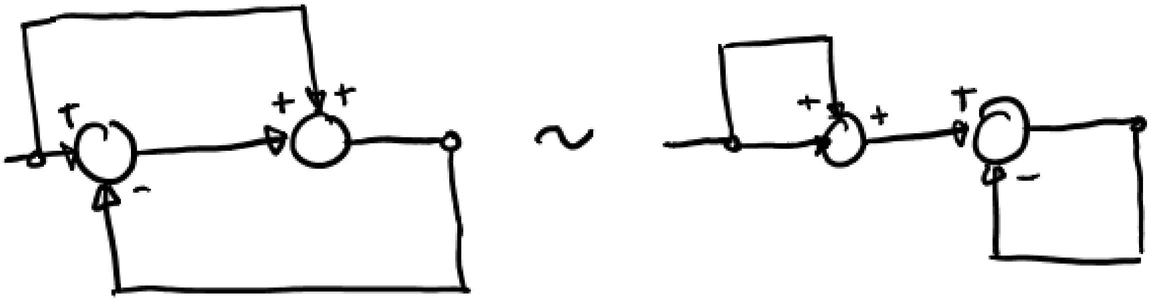


Rule 6: Interchanging Summing-points



When manipulating block diagrams we are for the most trying to manipulate the system in a way that Rule 5 can be used. This is because this is the rule that simplify the system the most.

If there is nothing in between two summing point, we are allowed to swap their places in the diagram.



As you can see, we can now use Rule 5 to close both the loops.

When we are constructiong block diagrams for differential equations, we usually start with the output (to the right), and work our way to the input (to the left).

Designing Block Diagrams

…..

1st Order Systems

Standar Form

$r_i \cdot e^{r_i}$, for real roots.

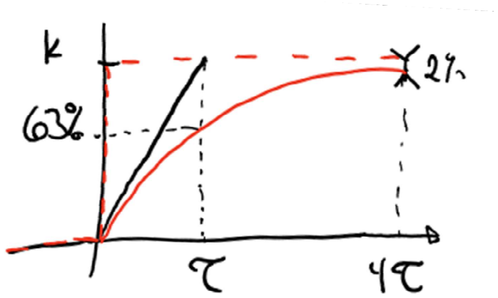
$k_i \cdot e^{r_i} \cdot \sin(bt + \varphi)$, $s = a \pm jb$, for complex roots.

$\frac{C(s)}{R(S)} = \frac{k}{\tau s+1}$, the standard transfer function.

If we have a step input, $R(s) = u(t) = \frac{1}{s}$;

$$C(s) = \frac{k}{\tau s + 1} \cdot \frac{1}{s} = \frac{k}{s} + \frac{-k}{s + \frac{1}{\tau}}$$

$$c(t) = k\left(1 + e^{-\frac{1}{\tau}t}\right)$$



If we take the ramp of the very start of $c(t)$ we can find τ
(time variable) by looking at where the ramp crosses the initial step input.

This will also always be where $c(t)$ is 63% close to the step input.

If we look at $c(t)$ after 4τ we know it will only have an error of 2% from the initial step input.

2^{nd} Order Systems

Standard Form

$$G(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta s\omega_n + \omega_n^2}$$

ζ: Damping ratio.

ω_n: Natural frequency.

Roots

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

1. $\zeta > 1$: Roots are negative, real poles.
2. $\zeta = 1$: Double real roots.
3. $\zeta = 0$: $s = \pm j\omega_n$ (Zero damping).
4. $\zeta < 0$: Positive real poles, system is unstable.
5. $0 < \zeta < 1$: Roots: $s = -\zeta\omega_n \pm j\omega_n \cdot \sqrt{1 - \zeta^2}$

$$\omega_d = j\omega_n \cdot \sqrt{1 - \zeta^2}$$

Step Input

$k \cdot u(t)$

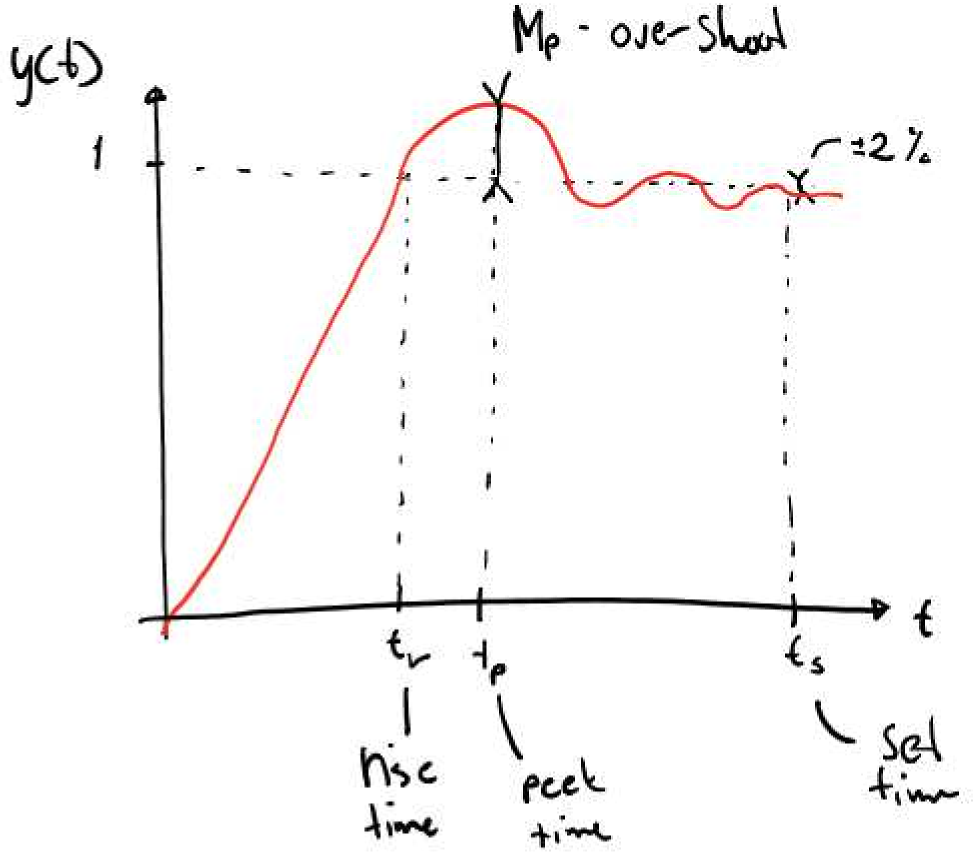
$$y(t) = k\left(1 - \frac{e^{-\zeta\omega_n}}{\sqrt{1 - \zeta^2}} \cdot \sin(\omega_d \cdot t + \varphi)\right)$$

$\tan\varphi = \frac{\sqrt{1-\zeta^2}}{\zeta}$, $\tau = \frac{1}{\zeta\omega_n}$, time constant

Final Value Theorem

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s \cdot E(s)$$

Transient Response Specification



General form:

$$G_C(s) = \frac{\omega_n^2}{s^2 + 2\zeta s\omega_n + \omega_n^2}$$

Peek Time:

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \cdot \sqrt{1-\zeta^2}}$$

Rise Time:

$$t_r = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}{\omega_n \cdot \sqrt{1-\zeta^2}}$$

(Radians)

Over Shoot:

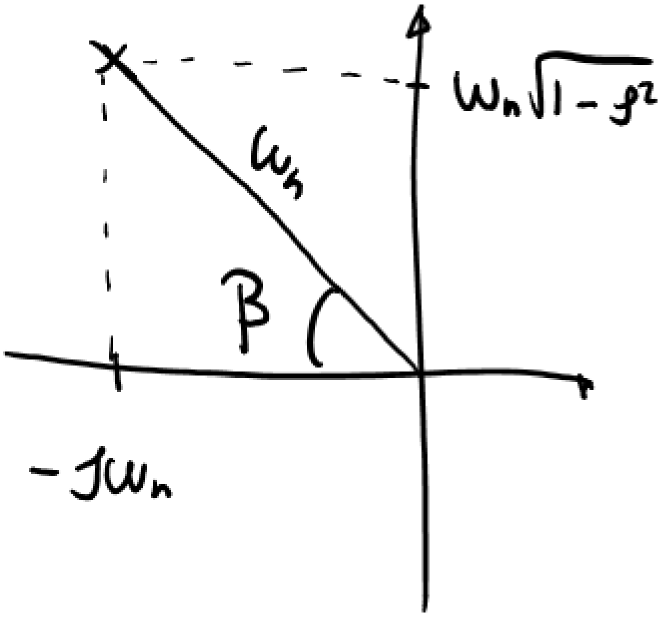
$$M_p = K\left(1 + e^{-\frac{\zeta \cdot \pi}{\sqrt{1-\zeta^2}}}\right)$$

Set Time:

$$T = \frac{1}{\zeta \cdot \omega_n}$$

$$\pm 2\% \rightarrow t_s = 4T = \frac{4}{\zeta \cdot \omega_n}$$

$$\pm 5\% \rightarrow t_s = 3T = \frac{3}{\zeta \cdot \omega_n}$$



$\cos\beta = \zeta$

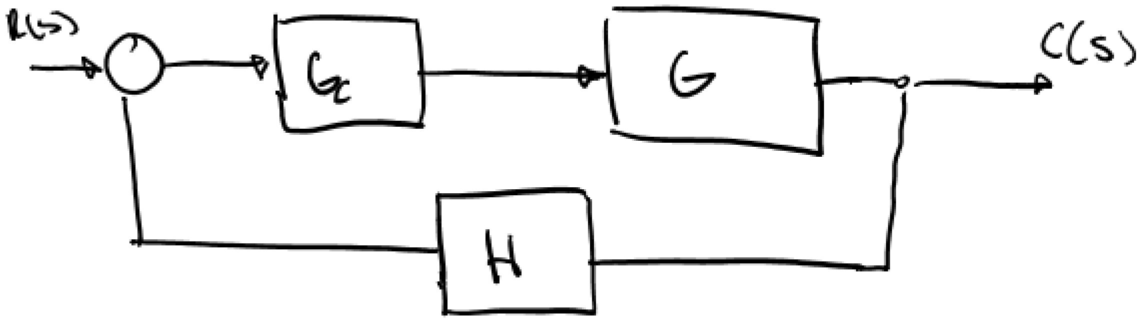
- Poles closed to the origin, means that the system has a slow response time.
- Small β means that the system has a large damping.
- The poles close to the imaginary axis $[Im]$ are called dominating poles.

No overshoot mean that the system will not oscillate.

Stability & Steady-state

Open- & Closed Loop Transfer Functions

Given the following system (this is a generalisation):



Open-loop Transfer Functions

$G_{OL}(s) = G_c(s) \cdot G(s) \cdot H(s)$

Closed-loop Transfer Functions

$$G_{CL}(s) = \frac{C(s)}{R(s)} = \frac{G_c(s) \cdot G(s)}{1 + G_c(s) \cdot G(s) \cdot H(s)}$$

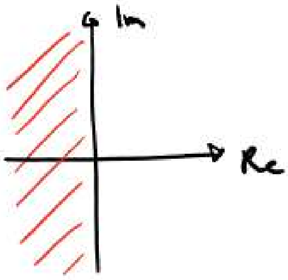
(Block diagram: Rule 5)

Zeros and Poles

Zeros: Roots in the numerator

Poles: Roots in the denominator

For a system to be stable all the poles have to be on the left side of the s-plane.



Steady-state analyse

We the *Final Value Theorem* we get that:

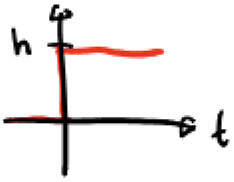
$$e_{r,ss} = \lim_{t\to\infty} e_r(t) = \lim_{s\to 0} s \cdot E_r(s) \cdot R(s)$$

$$e_{d,ss} = \lim_{t\to\infty} e_d(t) = \lim_{s\to 0} s \cdot E_d(s) \cdot D(s)$$

Input Functions

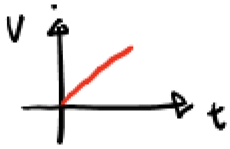
Step-input:

$$r(t) = h => R(s) = \frac{h}{s}$$



Ramp-input:

$$r(t) = v \cdot t => R(s) = \frac{v}{s^2}$$



Acceleration-input:

$$r(t) = \frac{a}{2} \cdot t^2 => R(s) = \frac{a}{s^3}$$

General:

$$r(t) = c \cdot \frac{t^n}{n!} => R(s) = \frac{c}{s^{n+1}}$$

Error table:

$$e_{r,ss} = \lim_{s \to 0} s \cdot \frac{1}{1 + \frac{K}{s^n}} \cdot \frac{c}{s^{n+1}}$$

Input	$N = 0$	$N = 1$	$N = 2$
Step $n = 0, \qquad c = h$	$\frac{h}{1 + k_h}$	0	0
Ramp $n = 1, \qquad c = v$	∞	$\frac{v}{k_v}$	0
Acceleration $n = 2, \qquad c = a$	∞	∞	$\frac{a}{k_a}$

$$k_h = \lim_{s \to 0} G_c(s) \cdot G_p(s)$$

$$k_v = \lim_{s \to 0} s \cdot G_c(s) \cdot G_p(s)$$

$$k_a = \lim_{s \to 0} s^2 \cdot G_c(s) \cdot G_p(s)$$

Routh's Stability Criterion

Routh's Array

$s(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s^1) = 0$

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_0 & & & &
\end{array}$$

$$\begin{array}{lcl}
b_1 & = & \dfrac{a_1 a_2 - a_0 a_3}{a_1} \\
b_2 & = & \dfrac{a_1 a_4 - a_0 a_5}{a_1} \\
b_3 & = & \dfrac{a_1 a_6 - a_0 a_7}{a_1} \\
 & \vdots &
\end{array}$$

The number of sign-changes in the first column indicates the number of poles in the right half plane. So if there are no sign-changes all the poles are in the left half plane, thus making the system stable.

A zero in the first column indicates a marginally stable system.

Ex.

$$Q(S) = 2s^4 + 4s^3 + 1s^2 + 2s + 2 = 0$$

Routh's array

$$\begin{array}{cc|ccc} & s^4 & 2 & 1 & 2 \\ & s^3 & 4 & 2 & 0 \\ 1 & s^2 & \varepsilon & 2 & \\ & s^1 & 2-\frac{8}{\varepsilon} & 0 & \\ 2 & s^0 & 2 & & \end{array}$$

$$b_1 = \frac{-1}{4}\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0 \rightarrow \text{replace with } \varepsilon$$

$$b_2 = \frac{-1}{4}\begin{vmatrix} 2 & 2 \\ 4 & 0 \end{vmatrix} = 2$$

$$c_1 = \frac{-1}{\varepsilon}\begin{vmatrix} 4 & 2 \\ \varepsilon & 2 \end{vmatrix} = 2 - \frac{8}{\varepsilon}$$

$$d_1 = \frac{-1}{2-\frac{8}{\varepsilon}}\begin{vmatrix} \varepsilon & 2 \\ 2-\frac{8}{\varepsilon} & 0 \end{vmatrix} = 2$$

Evaluate $\varepsilon \rightarrow 0$

$\Downarrow$

$$2 - \frac{8}{\varepsilon} = -\infty$$

then we get 2 sign-changes

1. positive → negative
2. negative → positive

2 poles in the right half plane

makes the system unstable.

Root Locus

Rules for Root Locus Development

Rule 1

The *root locus* is symmetrical with respect to the real axis.

Rule 2

The root locus originates on the poles of $G(s)H(s)$ (for $K = 0$) and terminates on the zeros of $G(s)H(s)$ (as $K \to \infty$), including those at infinity.

Rule 3

If the open-loop function had α zeros at infinity $\alpha \geq 1$, the root locus will approach α asymptote as K approaches infinity. The asymptotes are located at the angles

$$\theta = \frac{r \cdot 180°}{\alpha} \qquad r = \pm 1, \pm 3, \pm 5, \ldots$$

and those asymptotes intersect the real axis at the point

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

Rule 4

The root locus includes all point on the real axis to the left of an odd number of real critical frequencies (poles and zeros).

Rule 5

The breakaway points on a root locus will appear among the roots of the polynomial obtained for either

$$\frac{d[G(s)H(s)]}{ds} = 0$$

or, equivalently,

$$N(s)D'(s) - N'(s)D(s) = 0$$

when $N(s)$ and $D(s)$ are the numerator and dominator polynomials, respectively, of $G(s)H(s)$.

Rule 6

Loci will depart from a pole p_j (arrive at a zero z_j) of $G(s)H(s)$ at the angle $\theta_d(\theta_a)$, where

$$\theta_d = \sum_{i} \theta_{zi} - \sum_{\substack{i \\ i \neq j}} \theta_{pi} + (r \cdot 180°)$$

$$\theta_d = \sum_{i} \theta_{pi} - \sum_{\substack{i \\ i \neq j}} \theta_{zi} + (r \cdot 180°)$$

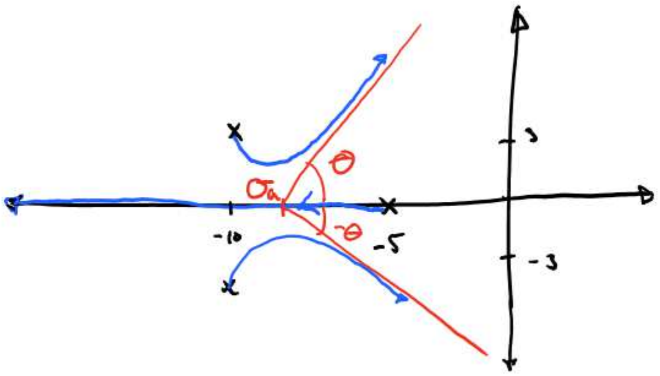
and where $r = \pm 1, \pm 3, \ldots$ and $\theta_{pi}(\theta_{zi})$ represent the angles from pole p_i (zero z_i), respectively, to $p_j(z_j)$.

Ex II.

$$G_0(s) = \frac{3}{(s+5)(s+10+3i)(s+10-3i)} = \frac{3}{s^3+25s^2+209s+545}$$

poles: $s = -5$

$s = -10 \pm j3$



$\alpha = 3$, 3 zeros in infinity

Rule 3:

$$\theta = \frac{r \cdot 180^\circ}{\alpha} = r \cdot 60^\circ, \quad r = \pm 1, \pm 3$$

$$\theta = \pm 60^\circ, 180^\circ$$

$$\sigma_a = \frac{(-5-10+j3-10-j3)-(0)}{3-0} = \frac{-25}{3} = -8{,}33$$

Lead Compensator

General:

$$G_c(s) = K \cdot \frac{T_1}{T_2} \cdot \frac{s + \frac{1}{T_1}}{s + \frac{1}{T_2}} = K \cdot \frac{T_1 \cdot s + 1}{T_2 \cdot s + 2}$$

Ex.

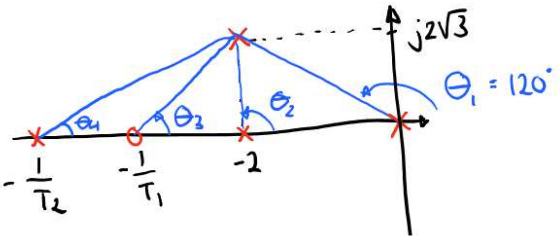
We have a system with a plant:

$$G_p(s) = \frac{5}{s(0.5s + 1)}$$

Let us say we want to create a lead compensator so the closed loop pole is located in

$$s = -2 \pm j2\sqrt{3}$$

First, we use the *angle criterion*:



$$\theta_3 - \theta_1 - \theta_2 - \theta_4 = \pm 180°$$

We can now move $-\frac{1}{T_1}$ so that $\theta_3 = \theta_2$.

Because poles and zeros cancel each other out we then get

$$-\theta_1 - \theta_4 = \pm 180°$$

$$120° - \theta_4 = \pm 180°$$

$$\theta_4 = 60°$$

By using the geometric of the triangle, we now know that

$$-\frac{1}{T_2} = -4 => T_2 = \frac{1}{4},$$

and as previously defined

$$-\frac{1}{T_1} = -2 => T_1 = \frac{1}{2}$$

This means that we now have

$$G_c(s) = K \cdot \frac{0.5}{0.25} \cdot \frac{s + 2}{s + 4} = 2K \cdot \frac{s + 2}{s + 4}$$

Now we can use the *magnitude criterion*:

$$\left|2K\cdot\frac{s+2}{s+4}\cdot\frac{5}{s(0.5s+1)}\right|_{s=-2+j2\sqrt{3}}=1$$

$$K=\left|\frac{s(s+4)}{20}\right|_{s=-2+j2\sqrt{3}}=0.8$$

$$G_c(s)=1.6\cdot\frac{s+2}{s+4}$$

Uncompensated system:

$$\frac{C(s)}{R(s)}=\frac{10}{s^2+2s+10}$$

And the compensated system is now

$$G_{OL}(s)=1.6\cdot\frac{s+2}{s+4}\cdot\frac{10}{s(s+2)}=\frac{16}{s(s+4)}$$

$$\frac{C(s)}{R(s)}=\frac{16}{s^2+4s+16}$$

Lag Controller

General:

$$G_c(s) = K \cdot \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}}, \qquad \beta > 1$$

With a lag controller, we can change the gain K without changing the transient characteristics.

To do this we want poles and zeros close to each other and to the origin.

Ex.

Bode Diagrams

Frequency Response of Systems

Steady-state response of systems to a sinusoidal input is called a *frequency response.*

$$r(t) = A \cdot \cos(\omega_i \cdot t) \;\rightarrow\; LTI \;\rightarrow c(t) = A \cdot |G(j\omega_i)| \cdot \cos(\omega_i \cdot t \cdot \varphi)$$

$$|G(j\omega_i| = |G(s)|_{s=j\omega}$$

$$\varphi(\omega_i) = \angle G(j\omega_i)$$

Ex.

$G(s) = \frac{5}{s+2}$; input: $r(t) = 7 \cdot \cos(3t)$

$G(s)\Big|_{s=j3} = \frac{5}{2+j3}$; $|G(j3)| = \frac{|5|}{|2+j3|} = \frac{5}{\sqrt{2^3+3^2}} = \underline{1.387}$

$\angle G(j3) = \angle 5 - \angle 2+j3 = 0 - \tan^{-1}\left(\frac{3}{2}\right) = -56.3^\circ$

$$\underline{\underline{C_{ss}(t) = 7 \cdot 1.387 \cdot \cos(3t - 56.3^\circ)}}$$

Bode Diagrams

A Bode diagram consists of two plots:

- Magnitude (Gain) vs. frequency
- Phase shift vs. frequency

Decibels $[dB] = 20\log_{10}|G(j\omega)|$

General forms

$$G(s) = \frac{k(1 + s/\omega_3)}{(1 + s/\omega_1)(1 + s/\omega_2)}$$

ω_i are called breakaway frequencies

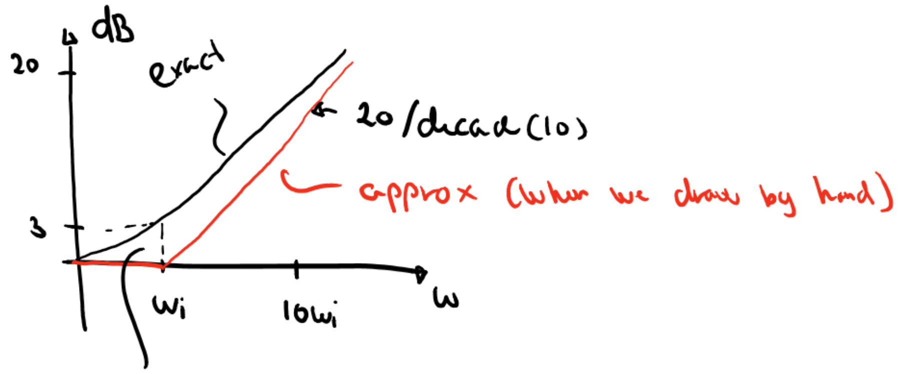
$$dB = 20\log\left|1 + j\frac{\omega}{\omega_i}\right| = 20\log\sqrt{1 + \left(\frac{\omega}{\omega_i}\right)^2}$$

$\omega = \omega_i$: $\quad dB = 20\log\sqrt{2} \quad = \quad 3.01\ dB$

$\omega = 10\omega_i$: $\quad dB = 20\log\sqrt{101} = 20.04\ dB$

If we increase with a factor of 10, we just add 20 to the dB.

$\omega \ll \omega_i = 20\log 1 = 0$



exact

20/decade(10)

approx (when we draw by hand)

error will be 3dB

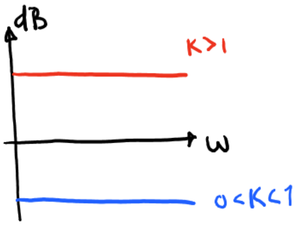
1. Constant Gain

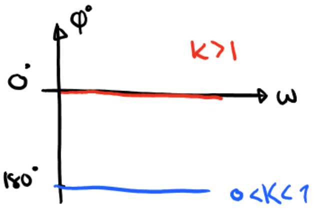
$G(s) = K$

$dB = 20 \log|K|$

$K > 1$: $\varphi = 0°$

$0 < K < 1$: $\varphi = -180°$



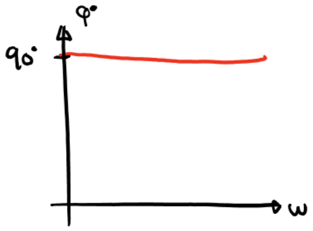


2. Poles and Zeros at Origin

Zero:

$G(s) = s$

$dB = 20 \log|j\omega| = 20 \log \omega$

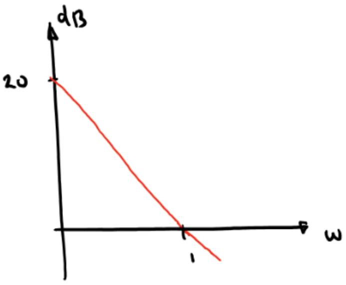


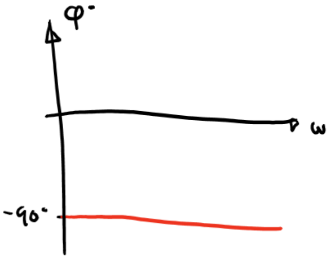
Pole:

$G(s) = \frac{1}{s}$

$dB = 20 \log\left|\frac{1}{j\omega}\right|$

$= 20 \log 1 - 20 \log|j\omega|$





3. Nonzeros and Nonpoles

Zero:

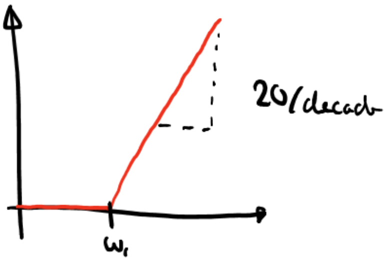
$G(s) = 1 + \frac{s}{\omega}$

Magnitude: $dB = 20\ \log\left|1 + j\frac{\omega}{\omega_i}\right| = 20 \log\sqrt{1 + \left(\frac{\omega}{\omega_i}\right)^2}$

$\omega = 0.1\omega_i$: $20 \log\sqrt{1.01} \approx 0 \quad dB$

$\omega = \omega_i$: $20 \log\sqrt{2} \quad \approx 3 \quad dB$

$\omega = 10\omega_i$: $20 \log\sqrt{101} \approx 20 \quad dB$

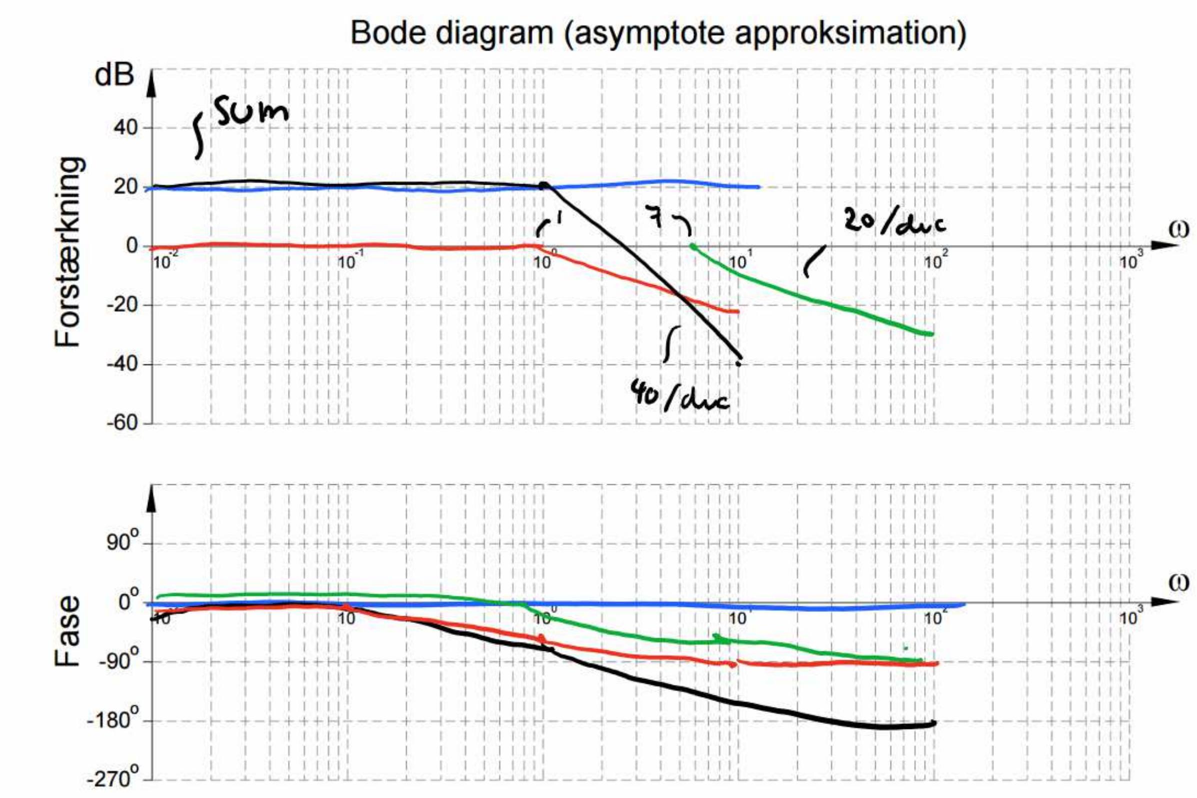


Phase shift:

$$1 + \frac{s}{\omega_i}\bigg|_{s=j\omega} = 1 + \frac{j\omega}{\omega_i} => \varphi(\omega) = \tan^{-1}\left(\frac{\omega}{\omega_i}\right)$$

Ex.

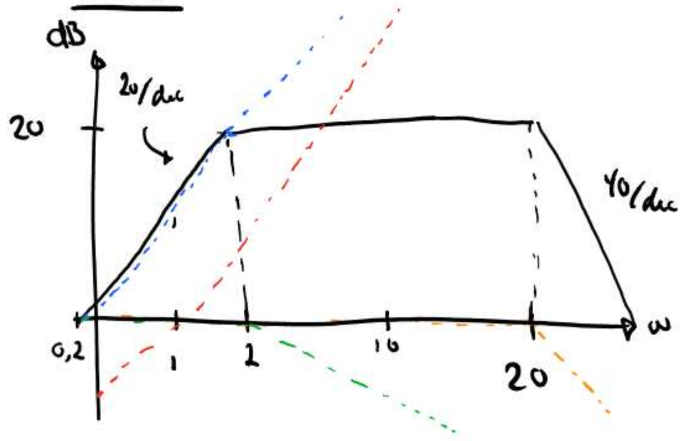
$$G(s) = \frac{10}{(s+1)\left(\frac{1}{7}s+1\right)}$$



Ex.II

Given the bode diagram (see drawing), find the function

15.6



$G(s) = ?$; $G_1(s) = s$ $\quad G_2(s) = \frac{1}{1+s/2}$

$+$

$G_3(s) = \frac{k \cdot s}{1+s/2}$ $\quad 2 < \omega < 20$

$G_4(s) = \frac{1}{(1+s/20)^2}$

$20 \log |G(j\omega)|_{\omega=2} = 20$

$20 \log |k j\omega|_{\omega=2} = 20$

$\log |2k| = 1 \Rightarrow K = 5$

$$G(s) = \frac{5 \cdot s}{(1+s/2)(1+s/20)^2}$$

Complex Poles and Zeros

General:

$$G(j\omega) = \frac{\omega_n^2}{s^2 + 2\,\zeta\omega_n s + \omega_n^2}\bigg|_{s=j\omega} = \frac{1}{1 + 2\zeta\left(j\dfrac{\omega}{\omega_n}\right) + \left(j\dfrac{\omega}{\omega_n}\right)^2}$$

Magnitude:

$$dB = 20\log|G(j\omega)| = -20\log\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

$$\omega \ll \omega_n: \quad dB = -20\log 1 = 0$$

$$\omega \gg \omega_n: \quad db = -20\log\frac{\omega^2}{\omega_n^2} = -40\log\frac{\omega}{\omega_n}$$

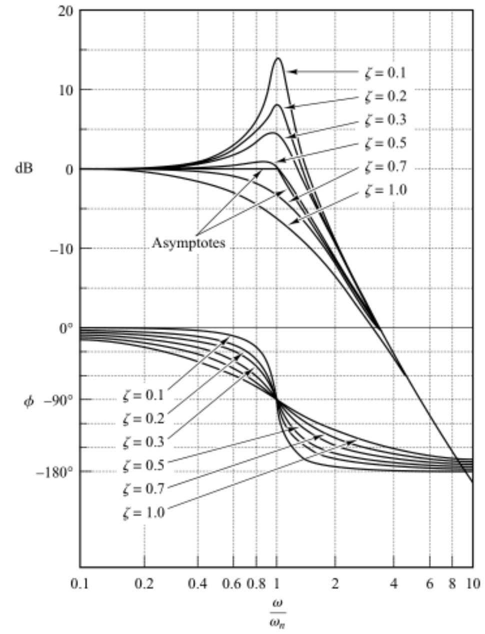
Phase Angle:

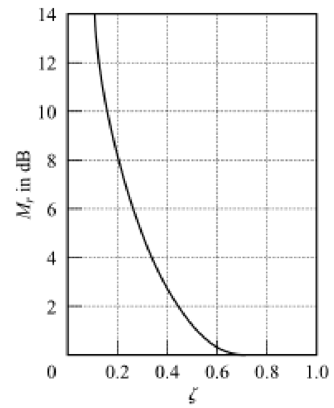
$$\varphi(\omega) = \angle\frac{1}{1 + 2\zeta\left(j\dfrac{\omega}{\omega_n}\right) + \left(j\dfrac{\omega}{\omega_n}\right)^2} = \tan^{-1}\left(\frac{2\zeta\left(\dfrac{\omega}{\omega_n}\right)}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2}\right)$$

$$\omega = 0 => \varphi(\omega) = 0°$$

$$\omega = \omega_n => \varphi(\omega) = -90°$$

$$\omega = \infty => \varphi(\omega) = -180°$$

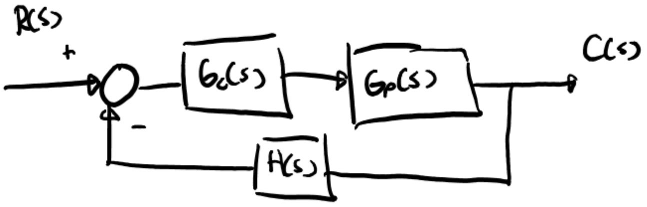




Controllers

Gain Compensator

Ex.



$$G_p(s)\cdot H(s)=\frac{4}{s(s+1)(s+2)};\qquad H(s)=1$$

Read Phase Margin:

$$20\log\left|G_p(j\omega)\right|=0\ dB\ \ =>\ \ \varphi=-168°\quad \omega=1.14\frac{rad}{s}$$

$PM=12°$, this should be $30°-60°$. Thus the system is very oscillatory.

$GM=3.52\ dB$, where $\varphi=-180°$, too small.

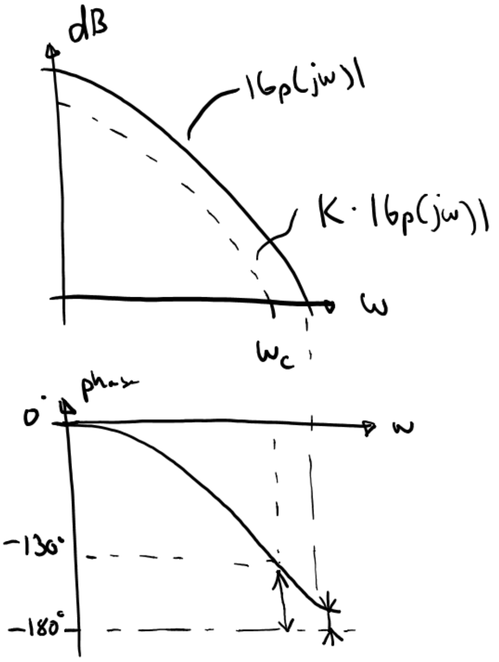
We want the system to have a $PM=50°$. To atchive this we have to change the gain K.

$K\cdot|G(j\omega)|=1\ \angle 130°$, from table given in the book (or any task) we find that

$\omega=\omega_c=0.5$, where ω_c is the crossover frequency.

$|G(j\omega)|=3.47$

$K=\frac{1}{3.47}$



Controller can not change angle of open loop

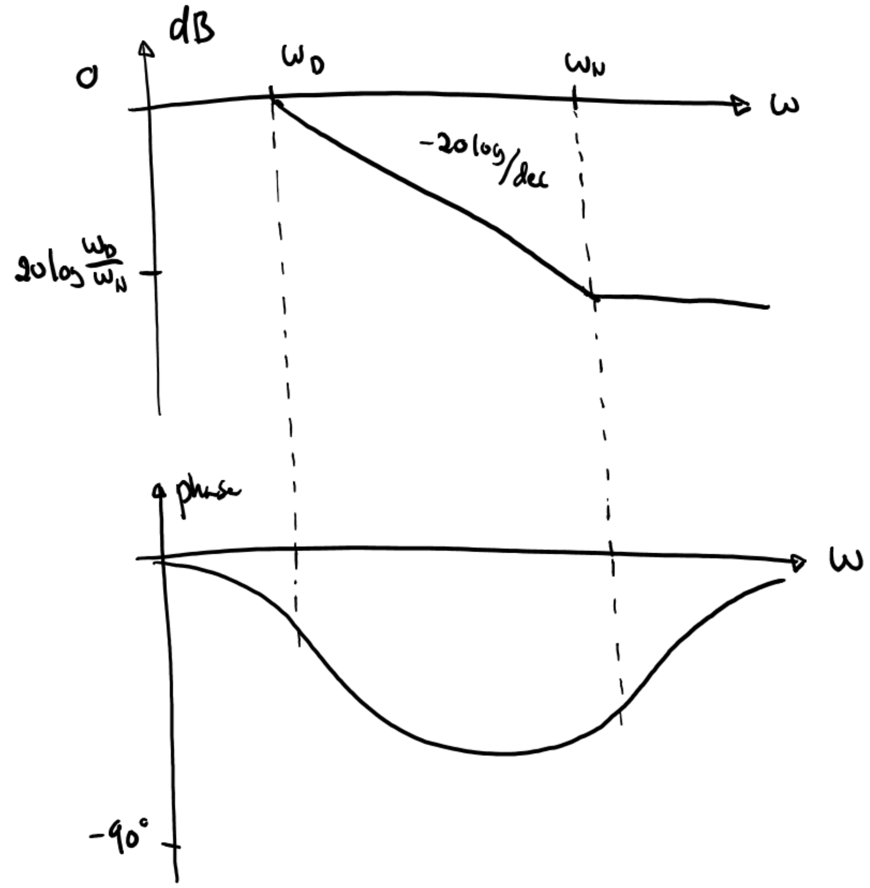
Phase Lag Compensator

Controller:

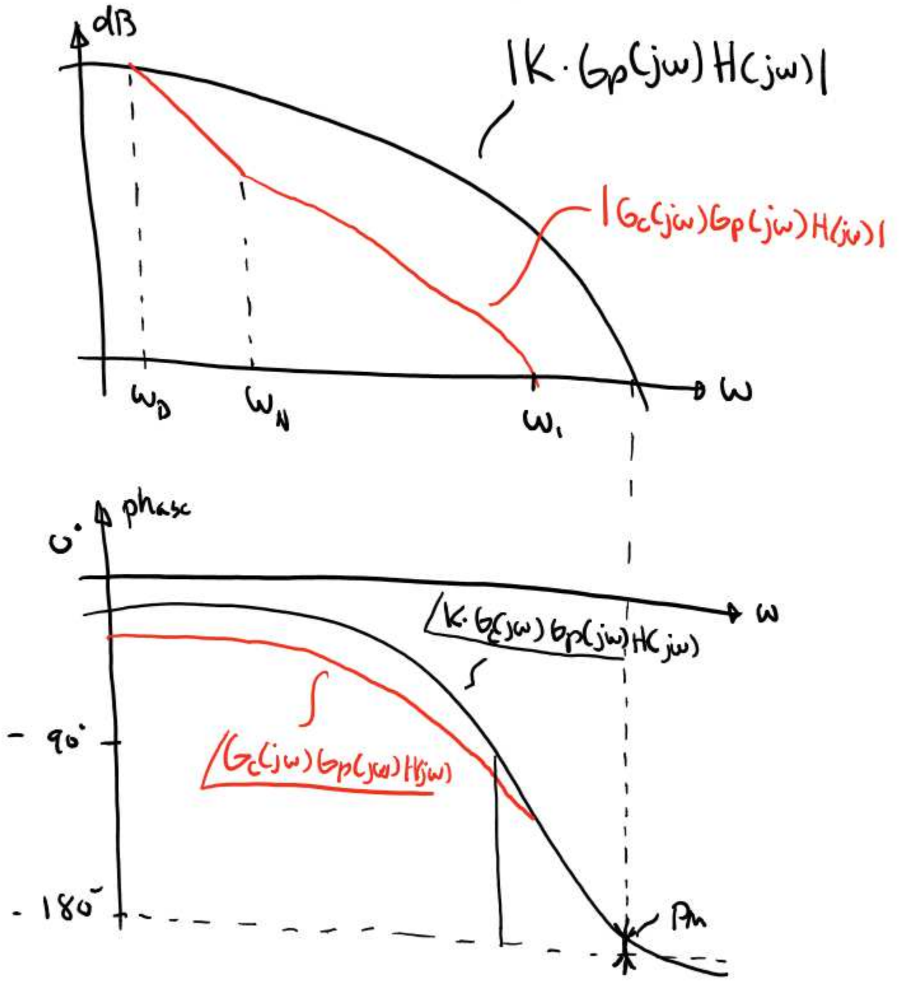
$$G_c(s) = K_c \cdot \beta \cdot \frac{Ts+1}{\beta Ts+1} = K \cdot \frac{1+s/\frac{1}{T}}{1+s/\frac{1}{\beta T}}$$

$$\frac{1}{T} = \omega_N, \qquad \frac{1}{\beta T} = \omega_D$$

Bode Plot for the compensator, $K = 1$:



Bode Plot for the control sytem:



Design Procedure

$K = K_c \cdot \beta$

$$G_c(s) = K \cdot \frac{1 + s/\frac{1}{T}}{1 + s/\frac{1}{\beta T}}$$

$$\frac{1}{T} = \omega_N, \qquad \frac{1}{\beta T} = \omega_D$$

1. Adjust the gain K to satisfy low frequency specifications (steady state error; step, ramp, parabolic inputs).

2. Find the frequency ω_1 at which the angle of $K \cdot G_p(j\omega) \cdot H(j\omega) = -180° + PM + 5°$

3. Magnitude of zero: $\frac{1}{T} = \omega_n = 0.1 \cdot \omega_1$

4. At ω_1 read: $\left|G_c(j\omega_1) \cdot G_p(j\omega_1) \cdot H(j\omega_1)\right| = 1$

$$\beta = K \cdot G_p(j\omega_1) \cdot H(j\omega_1)$$

5. Calculate pole: $\frac{1}{\beta T} = \omega_0$ and $K_c = \frac{K}{\beta}$

6. Compensator transfer function is $G_c(s) = K \cdot \frac{s+\omega_N}{s+\omega_D}$

Ex.

$G_p(s) \cdot H(s) = \frac{4}{s(s+1)(s+2)}$; $e(t)$ for ramp input shold be $e(t) = \frac{1}{2}$, PM = 50°

1. step: type 1 system and ramp input: $e(t) = \frac{1}{K_v}$

1. approach: $\frac{4 \cdot K}{s(s+1)(s+2)} = \frac{2 \cdot K}{s(1+2)(1+s/2)}$

$$e(t) = \frac{1}{K_v} = \frac{1}{2} = \frac{1}{2K} \Rightarrow K = 1$$

2. approach: $K_v = \lim_{s \to 0} s \cdot K \cdot G_p(s) \cdot H(s) = \frac{2}{1} = 2K \Rightarrow K = 1$

2. step: $\angle K \cdot G_p(j\omega_1) \cdot H(j\omega_1) = -180° + 50° + 5° = -125°$

from table $\omega_1 = 0{,}4$ for $\varphi = -123.5°$, close enough.

3. step: $\frac{1}{T} = 0.1\omega_1 = 0{,}04$

4. step: $\beta = |K \cdot G_p(j\omega_1) H(j\omega_1)| = 4.55$

5. step: $\frac{1}{\beta T} = \frac{0{,}04}{4.55} = 0{,}0088$

6. step: $$\underline{\underline{G_c = \frac{1}{4.55} \cdot \frac{s + 0{,}04}{s + 0{,}0088}}}$$

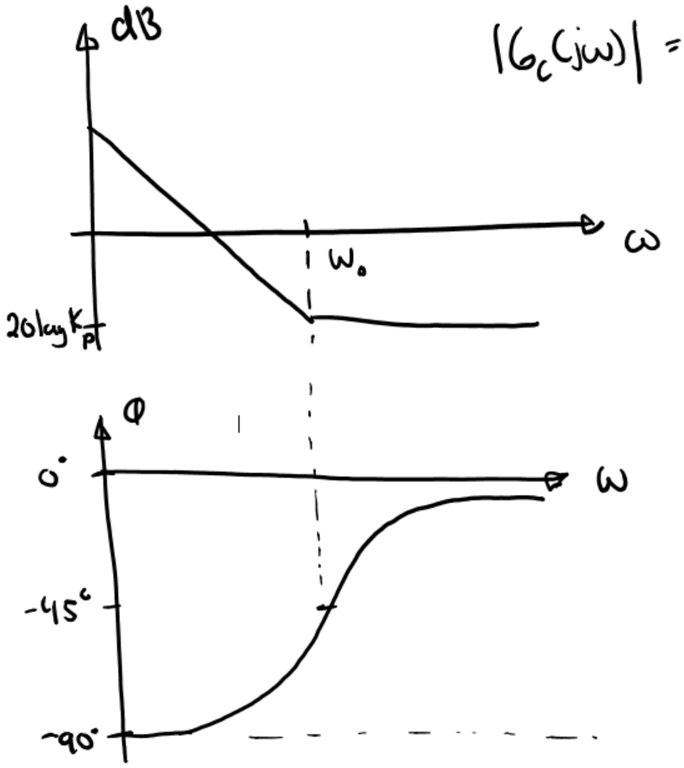
PI-Controller

$$G_c(s) = K_p\left(1 + \frac{1}{T_i \cdot s}\right) = \frac{K_I(1 + s/\omega_0)}{s}$$

$$\omega_0 = \frac{K_I}{K_p}$$

Bode Plot:

$$|G_c(j\omega)| = \left|K_p + \frac{K_I}{j\omega}\right|$$



Design Procedure

1. Adjust the DC-gain of $G_p(s) \cdot H(s)$ by a factor of K_c to satisfy low frequency specifications.

2. Find the frequency ω_1 at which the angle of $G_p(j\omega_1) \cdot H(j\omega_1) = -180° + PM + 5°$

3. $K_p = \frac{1}{|K_c \cdot G_p(j\omega_1) \cdot H(j\omega_1)|}$

4. Magnitude of zero: $\omega_0 = \frac{K_I}{K_p} = 0.1 \cdot \omega_1$ and $K_I = 0.1 \cdot \omega_1 \cdot K_p$

5. Controller is then: $G_c(s) = K_c\left(K_p + \frac{K_I}{s}\right)$

Ex.

$$G_p(S)\,H(S) = \frac{4}{S(S+1)(S+2)} \quad ; \quad PM = 50°$$

1. $K_c = 1$

2. $\omega_1 = 0{,}4 \qquad \angle G_{OL}(S) = -125$ (from last Ex.

3. $K_p = \frac{1}{4{,}55} = 0{,}220$

4. $K_I = 0{,}1\,\omega_1 \cdot K_p = 0{,}1 \cdot 0{,}4 \cdot 0{,}220 = 0{,}0088$

5.

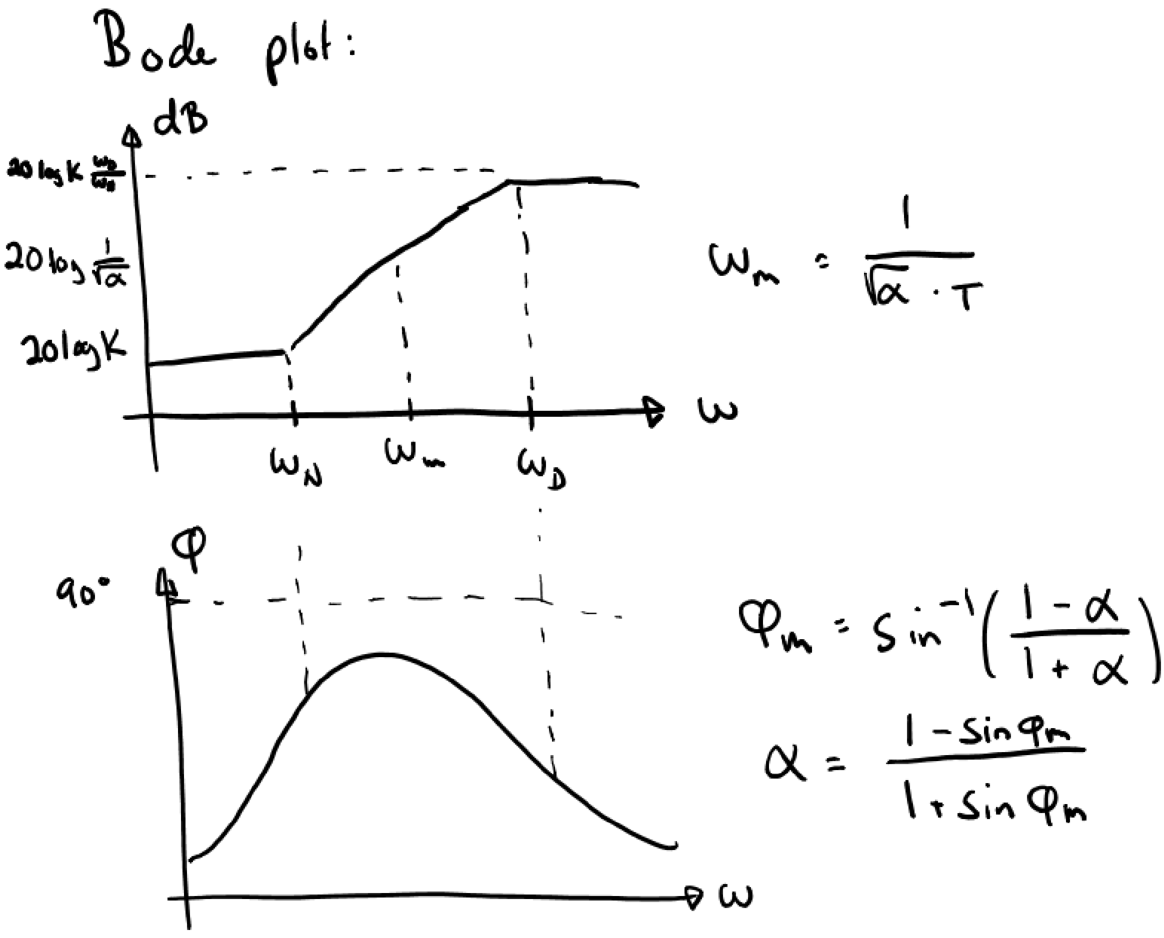
$$\underline{\underline{G_c(S) = 0{,}220 + \frac{0{,}0088}{S}}}$$

Phase Lead Controller

$$G_c(s) = K_c \cdot \alpha \cdot \frac{Ts + 1}{\alpha Ts + 1}, \qquad K = K_c \cdot \alpha$$

$$K \cdot \frac{1 + \frac{s}{\omega_N}}{1 + \frac{s}{\omega_D}}, \qquad \omega_N = \frac{1}{T}, \quad \omega_D = \frac{1}{\alpha T}$$

Bode Plot:



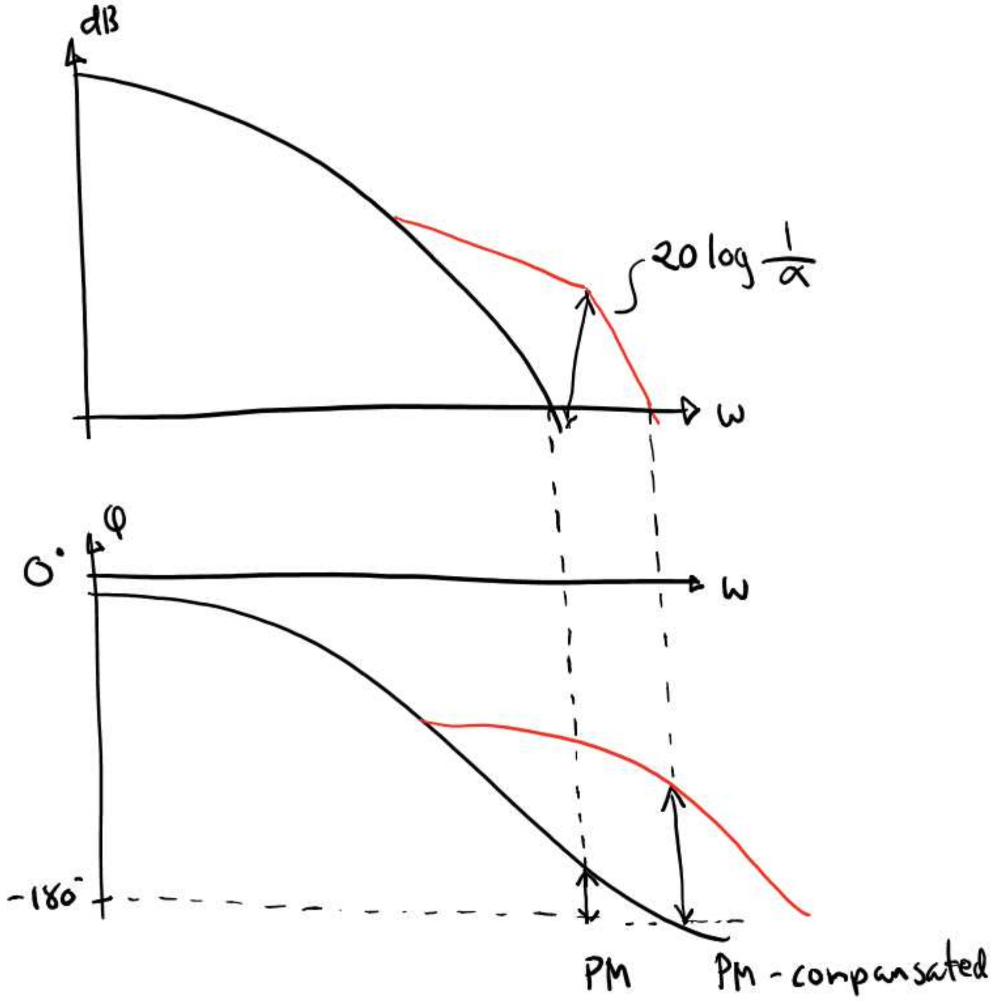
$$\omega_m = \frac{1}{\sqrt{\alpha} \cdot T}$$

$$\varphi_m = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)$$

$$\alpha = \frac{1 - \sin\varphi_m}{1 + \sin\varphi_m}$$

Bode Plot of the Plant:

Bode plot of the plant:



Ex.

$G_p(s) \cdot H(s) = \frac{4}{s(s+1)(s+2)}$; $\omega = 1.14$ $\varphi = -168°$

PM = 15° What we got

We want : PM = 50°, we need to add $50° - 12° = 38°$

$\varphi_m = 50° - 12 + 12° = \underline{50°}$ (lead controller)

usually 5°, but that was not enought

$$\alpha = \frac{1 - \sin 50°}{1 + \sin 50°} = 0{,}1325$$

gain added ω_m (yet unknown)

$$20 \log \frac{1}{\sqrt{\alpha}} = 20 \log \frac{1}{\sqrt{0{,}1325}} = 8{,}7 \text{ dB}$$

From table: $\omega = 1.41 \rightarrow -3{,}52$ dB

$\omega = 2.0 \rightarrow -10$ dB

look for $-8{,}7\text{dB} \rightarrow \omega_1 = 1{,}9$ dB : new ω_c

$$\omega_c = \omega_m = \frac{1}{\sqrt{\alpha} \cdot T} \Rightarrow \frac{1}{T} = \sqrt{\alpha} \cdot \omega_m = 0{,}6915$$

pole : $\frac{1}{\alpha T} = \frac{0{,}6915}{0{,}1325} = 5.2197$ $K_c = \frac{k}{\alpha} = 7.54$

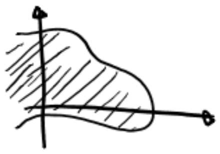
Controller : $\underline{\underline{G_c(s) = 7.54 \frac{s + 0{,}6915}{s + 5.2197} \cdot \frac{4}{s(s+1)(s+2)}}}$

Nyquist Analysis

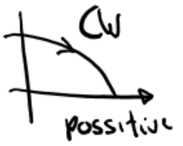
Definitions

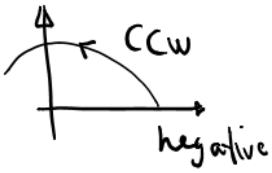
1. *Closed contour* (begin and ending in the same point-

2. All points inside the contour (or to the right) are said to be *enclosed* by it.



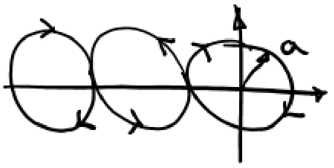
3. A clockwise (CW) traverse around the contour is defined as positive





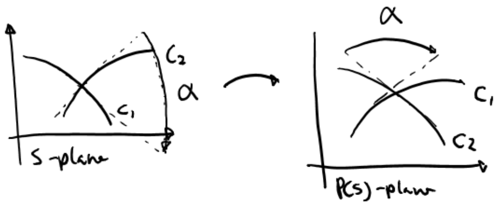
4. A closed contour in $P(s)$ plane is said to make n positive encirclements of the origin if a radial line is drawn from origin to a point on the $P(s)$-curve rotates CW $n \cdot 360°$ when going around the path.

Ex.
The vector a will only rotate around the origin $N = 1$ times.



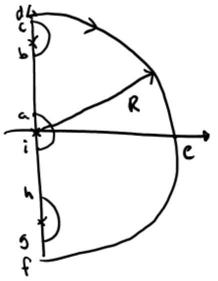
5. Closed contour in the s-plane maps into closed contour in the $P(s)$-plane

6. $P(s)$ is a conformal maping. This means angle and direction of interception of curves in the s-plane is preserved in the $P(s)$-plane.



7. Total number of encirclements N_0 of origin of closed contours $P(s)$ maped from closed contour in the s-plane, is equal to the numbers for zeros Z_0 minus the number of poles P_0 of $P(s)$ enclosed by the s-plane contour. ($N_0 = Z_0 - P_0$)

The Nyquist Path

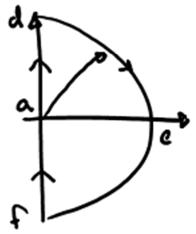


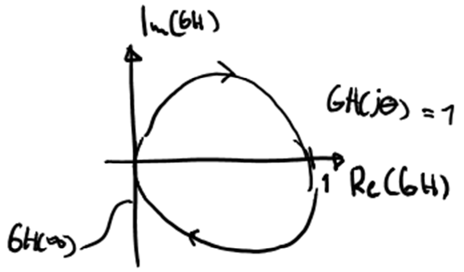
Every pole and zero of $P(s)$ in the right half plane is enclosed by the Nyquist path when maped into the $P(s)$-plane.

Ex.1

Type 0-system

$$P(s) = GH(s) = \frac{1}{s+1}$$





$$\text{Path}: \widehat{fad} : s = \lim_{R\to\infty} Re^{j\theta} \qquad 90^\circ \leq \theta \leq -90^\circ$$

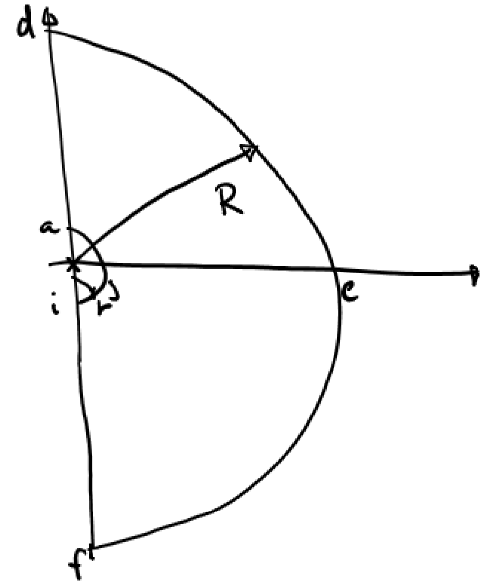
$$GH(s)\Big|_{\widehat{def}} = GH(\infty) = \frac{1}{\lim\limits_{R\to\infty} Re^{j\theta} + 1} = \lim_{R\to\infty}\left(\frac{1}{Re^{j\theta}+1}\right)$$

$$|GH(\infty)| = \lim_{R\to\infty}\left|\frac{1}{Re^{j\theta}+1}\right| \leq \lim_{R\to\infty}\left|\frac{1}{R-1}\right| = 0 \; ; |a+b| \geq \big||a|-|b|\big|$$

Semicircle maps into a point in origin.

Ex.2

Type 1: $GH(s) = \frac{1}{s(s+1)}$



Path : $\overset{\frown}{ad}$: $s = j\omega \quad 0 < \omega < \infty$

$$GH(j\omega) = \frac{1}{j\omega(j\omega+1)} = \frac{1}{\omega\sqrt{\omega^2+1}} \angle 90^\circ - \tan^{-1}$$

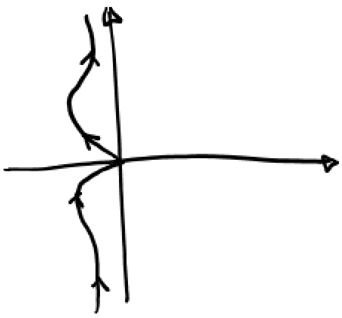
$$\lim_{\omega \to 0} GH(j\omega) = \infty \angle 90^\circ$$

$$\lim_{\omega \to \infty} GH(j\omega) = 0 \angle -180^\circ$$

$\overset{\frown}{ija}$: $s = \lim_{r \to 0} re^{j\theta} \qquad -90^\circ \leq \theta \leq 90^\circ$

$$\lim_{r \to 0} GH(re^{j\theta}) = \lim_{r \to 0} \left(\frac{1}{re^{j\theta}(re^{j\theta}+1)} \right)$$

$$= \lim_{r \to 0} \left(\frac{1}{re^{j\theta}} \right) = \infty e^{j\theta} = \infty \angle -0$$



Point i : $GH = \infty \angle 90^\circ$

Point j : $GH = \infty$

Point a : $GH = \infty \angle -90^\circ$

Note: Here we can also just plot $G \cdot H(j\omega)$ into a table for ω to find the values we need.

Nyquist Stability Criterion

Nyquist Stability Criterion states how many zeros of $1 + G \cdot H$ are in the right half plane. ($1 + G \cdot H's$ zeros are the poles of the closed loop transfer function $\frac{C(s)}{R(s)}$)

Theorem

$N = Z_0 - P_0$

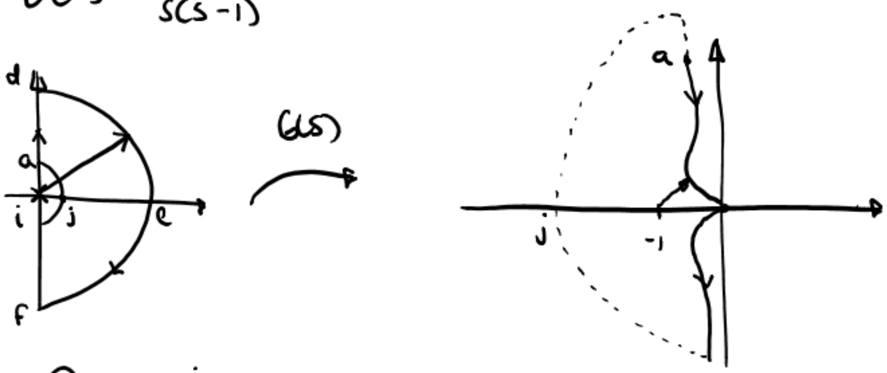
N: Number of CW encirclements of $(-1,0)$, this is because we have $\mathbf{1} + G \cdot H$.

Z_0: Numbre of zeros in the right half plane for $1 + G \cdot H$

P_0: Numbre of poles in the right half plane in open loop $G \cdot H$

Ex.

$$G(s) = \frac{1}{s(s-1)}$$



$$\widehat{ad} : s = j\omega \quad ; \quad 0 < \omega \leq \infty$$

$$G(j\omega) = \frac{1}{j\omega(j\omega - 1)} = \frac{1}{\omega\sqrt{\omega^2+1}} \cdot \underline{/-90^\circ - \tan^{-1}(-\omega)}$$

$$\lim_{\omega \to 0} G(j0) = \infty \underline{/-90^\circ - 180^\circ} = \infty \angle -270^\circ$$

$$\lim_{\omega \to \infty} G(j\infty) = 0 \underline{/-90^\circ - 90^\circ} = 0 \angle -180^\circ$$

$$\widehat{ija} \Rightarrow \lim_{r \to 0} GH(re^{j\theta}) = \lim_{r \to 0} \frac{1}{re^{j\theta}(re^{j\theta} - 1)}$$

$$= \lim_{r \to 0} \frac{1}{re^{j\theta}(-1)} = \lim_{r \to 0} \frac{-1}{r} e^{-j\theta}$$

$$= \infty \angle -180^\circ - \theta \quad ; \quad -90^\circ \leq \theta \leq 90^\circ$$

Relative Stability

Phase crossover frequency: $\omega\pi$

$GH(j\omega) = 180°$

Gain crossover frequency: ω_c

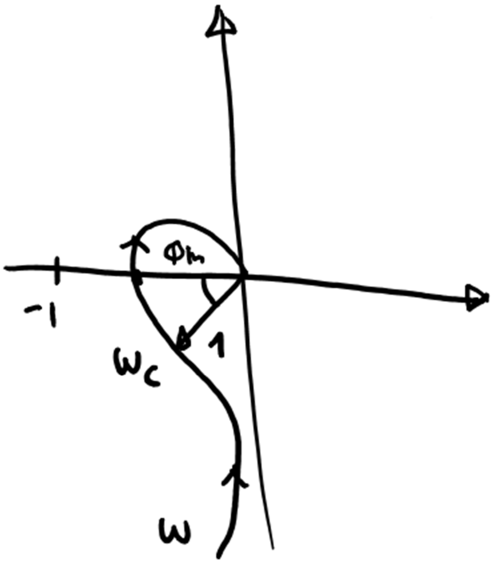
$|GH(j\omega\pi)| = 1$

Gain margin: G_M

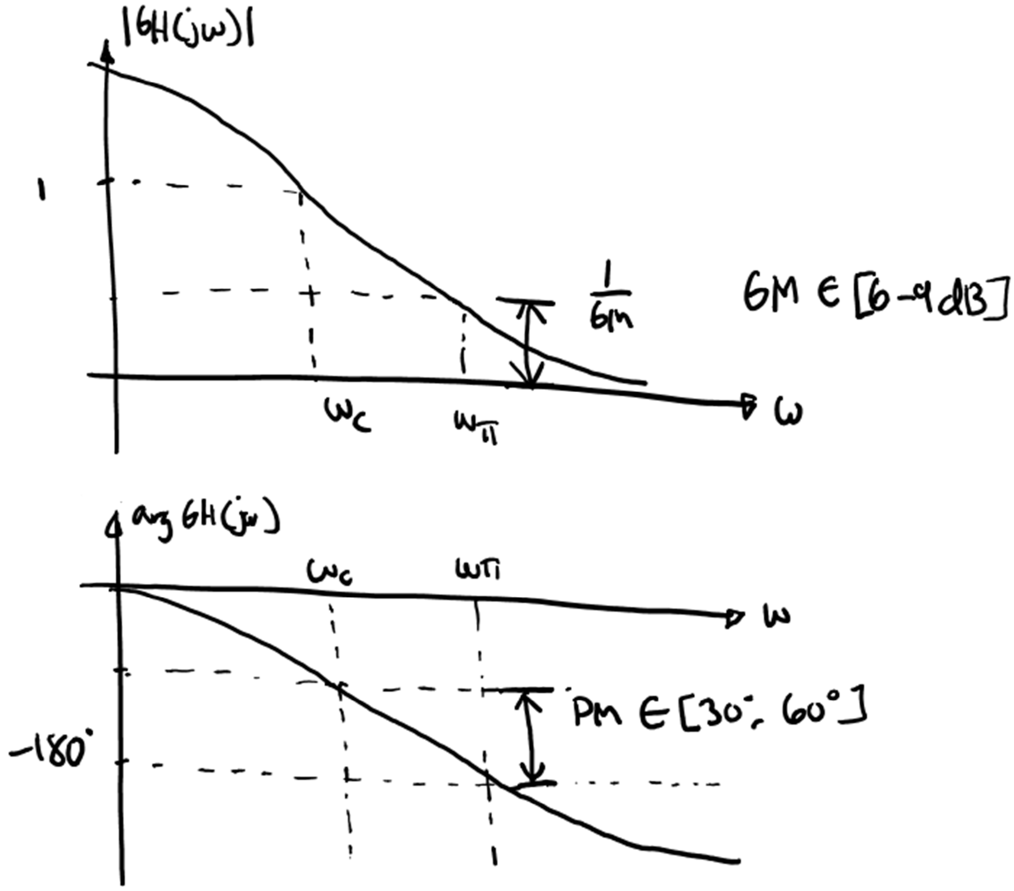
$K_M \cdot |GH(j\omega\pi)| = 1$

Phase Margin: P_M

$\varphi_M = (180° + \angle GH(j\omega_c)$



Relative Stability in Bode Plot



Note: For non-minimum-phase systems *do not* use Bode plot. Use Nyquist diagrams.

If ω_c is reduced:

- Bandwidth reduced
- Rise time increased
- Phase margin increased
- Damping increased
- Less overshoot

Control System Design

Phase Lead Controller

$$G_{c(s)} = K_c \cdot \alpha \cdot \frac{Ts+1}{\alpha Ts+1}; \quad 0 < \alpha < 1$$

Break frequency:

$$\omega_n = \frac{1}{T}, \quad \omega_0 = \frac{1}{\alpha T}$$

Bode form:

$$G_c(s) = K \cdot \frac{1 + s/\omega_n}{1 + s/\omega_0}$$

Magnitude:

$$|G_c(j\omega)| = |K_c \cdot \alpha| \cdot \frac{\sqrt{1+(\omega T)^2}}{\sqrt{1+(\alpha\omega T)^2}}$$

$$\omega \to 0: \quad |G_c(j\omega)| = K_c \cdot \alpha$$

$$\omega \to \infty: \quad |G_c(j\omega)| = |K_c \cdot \alpha| \cdot \frac{1}{\alpha}$$

Phase:

At a frequency,

$$\omega_m = \frac{1}{\sqrt{\alpha} \cdot T}$$

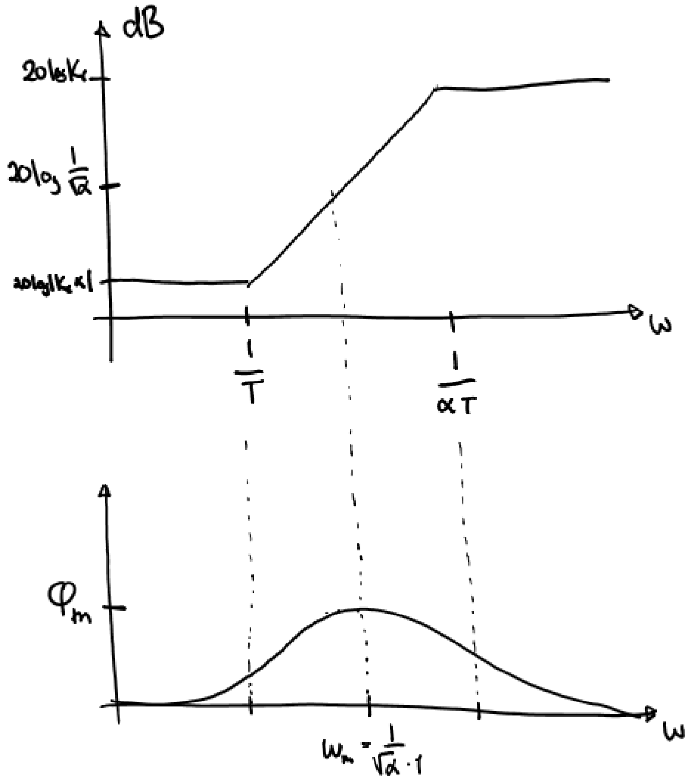
the phase shift is at its maximum with a value:

$$\sin \varphi_m = \frac{1-\alpha}{1+\alpha}$$

and a gain at

$$K_m = \frac{1}{\sqrt{\alpha}}$$

Bode Plot:



$$\alpha = \frac{1-\sin\varphi_m}{1+\sin\varphi_m}$$

Ex.: 7-26 from the book

$$G(s) = \frac{4}{s(s+2)} = \frac{2}{s(1+s/2)}$$

Performance Spec.:

$K_v = 20$, $PM > 50°$, $GM > 10\,dB$

Lead compensator: $G_c(s) = K_c \cdot \alpha \cdot \frac{Ts+1}{\alpha Ts+1}$; $0 < \alpha < 1$

Open loop: $G_c(s) \cdot G(s) = G_1(s) = K_c \cdot \alpha \cdot G(s) = K \cdot G(s)$

1. step: $K_v = \lim_{s \to 0} \cdot s \cdot G_c(s) \cdot G(s) = \lim_{s \to 0} s \cdot \frac{Ts+1}{\alpha Ts+1} \cdot G_1(s)$

$$= \lim_{s \to 0} s \cdot K \cdot \frac{2}{s(1+s/2)} = 2K = 20 \Rightarrow \underline{\underline{K=10}}$$

2. step: Bode plot of $G_1(j\omega) = \frac{20}{j\omega(1+j\omega/2)}$; $PM = 17^\circ$

$GM = \infty$

need to add: $33^\circ + 5^\circ \approx \underline{38^\circ}$

$$\alpha = \frac{1-\sin 38^\circ}{1+\sin 38^\circ} = 0{,}24 \; ; \; 20 \log \frac{1}{\sqrt{\alpha}} = 6.2\,dB$$

new crossover frequency

So gain (look for) $|G_1(j\omega)| = -6.2\,dB \Rightarrow \omega = 9 \frac{rad}{s}$

$$\omega_m = \frac{1}{\sqrt{\alpha} \cdot T} \Rightarrow T = \frac{1}{\sqrt{\alpha}\,\omega} \quad \text{or} \quad \frac{1}{T} = \sqrt{\alpha} \cdot \omega_c$$

$$\frac{1}{T} = \sqrt{0{,}24} \cdot 9 = 4.4 \qquad \frac{1}{\alpha T} = \frac{4.4}{0{,}25} = 18{,}4$$

$$K_c = \frac{K}{\alpha} = \frac{10}{0{,}24} = 41.7$$

$$\Rightarrow \underline{\underline{G_c(s) = 41.7 \cdot \frac{s+4.4}{s+18.4}}}$$

Laplace Table

	$f(t)$	$F(s)$
1	unit impulse $\delta(t)$	1
2	unit step $1(t)$	$\frac{1}{s}$
3	unit ramp t	$\frac{1}{s^2}$
4	e^{-at}	$\frac{1}{s+a}$
5	te^{-at}	$\frac{1}{(s+a)^2}$
6	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
8	$t^n \quad (n = 1,2,3,\dots)$	$\frac{n!}{s^{n+1}}$
9	$t^n e^{-at} \quad (n = 1,2,3,\dots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
11	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
12	$\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at} - ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$
13	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega}$
14	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega}$
15	$\frac{1}{a^2}(at-1+e^{-at})$	$\frac{1}{s^2(s+a)}$
16	$\frac{\omega_n}{\sqrt{1-\varsigma^2}}e^{-\varsigma\omega_n t}\sin \omega_n\sqrt{1-\varsigma^2}\,t$	$\frac{\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}$
17	$\frac{-1}{\sqrt{1-\varsigma^2}}e^{-\varsigma\omega_n t}\sin(\omega_n\sqrt{1-\varsigma^2}\,t-\phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\varsigma^2}}{\varsigma}$	$\frac{s}{s^2+2\varsigma\omega_n s+\omega_n^2}$
18	$1-\frac{1}{\sqrt{1-\varsigma^2}}e^{-\varsigma\omega_n t}\sin(\omega_n\sqrt{1-\varsigma^2}\,t+\phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\varsigma^2}}{\varsigma}$	$\frac{\omega_n^2}{s(s^2+2\varsigma\omega_n s+\omega_n^2)}$