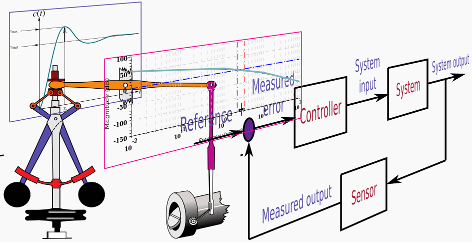
MAS107 Feedback Control Systems

Key Equations

$$G_c(s) = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\alpha T_1}\right)\left(s+\frac{\alpha}{T_2}\right)}$$



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Transfer Function

Transfer function of a system with input $R(s)$ and output $C(s)$

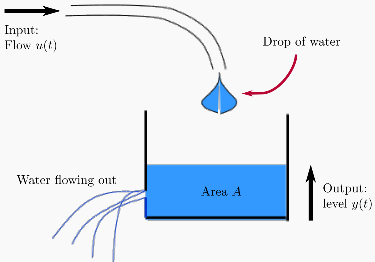
Transfer Function

$$G(s) = \frac{C(s)}{R(s)}$$

Final Value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Modeling of Water-tank Systems



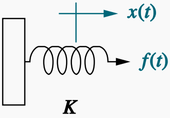
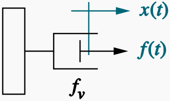
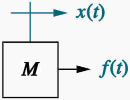
How do we model this?

- Mass balance equation: $\text{Flow}_{in} - \text{Flow}_{out} = \frac{d(\text{Volume})}{dt}$
- Equation: $u_{in} - u_{out} = A\frac{dy}{dt}$

Problem

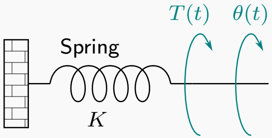
Given a system with several tanks, find the differential equations and Transfer function.

Translational Mechanical Systems

Component	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M \frac{d^2x(t)}{dt^2}$	Ms^2

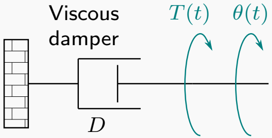
Rotational Mechanical Systems

- Torque replaces force.
- Angular displacement replaces translational displacement.
- Inertia replaces mass.



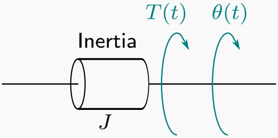
$$T(t) = K\theta(t)$$

K



$$T(t) = D\dot{\theta}(t)$$

$D\,s$



$$T(t) = J\ddot{\theta}(t)$$

$J\,s^2$

Finding the Equations of Motion for mechanical network

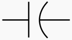
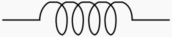
- **Method 1: Newton's law** [Sum of applied forces] $= M\frac{d^2x}{dt^2}$
  - Choose coordinates and directions
  - Draw the free-bod diagram for each inertial
  - Applying Newton's Law to find the equations for each motions
- **Method 2: Impedances** [Sum of applied forces] = [Sum of impedances] $X(s)$
  - Choose coordinates and directions
  - Find inpedances to each motion
  - Find impedances between motions
  - Applying the following equations

$$\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_1\end{bmatrix} X_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_2\end{bmatrix} X_2(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_1\end{bmatrix}$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_2\end{bmatrix} X_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_2\end{bmatrix} X_2(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_2\end{bmatrix}$$

Electrical Network: Impedances

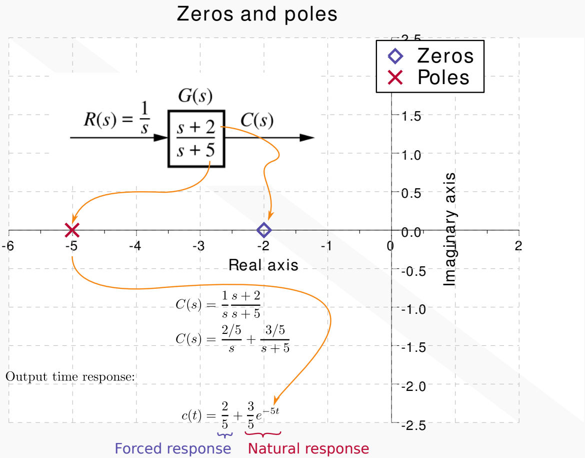
Voltage-current and impedance relationships for capacitors, resistors, and inductors

Component	**Voltage-current**	**Impedance** $Z(s) = V(s)/I(s)$	**Admittance** $Y(s) = I(s)/V(s)$
 Capacitor	$i(t) = C\frac{dv(t)}{dt}$	$\frac{1}{Cs}$	$C\,s$
 Resistor	$v(t) = Ri(t)$	R	$\frac{1}{R}$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	Ls	$\frac{1}{L\,s}$

Note: The following set of symbols and units is used: $v(t)$ – V (volts), $i(t)$ – A (amps), C – F (farads), R – Ω (ohms), L – H (henries).

Poles and Zeros

- Poles of a transfer functions:
  The values of s that cause the transfer function to become infinite.
- Zeros of a transfer functions:
  The values of s that cause the transfer function to become zero.



- Poles of the transfer function: Determine the form of the natural response.
- Poles of the input function: Determine the form of the forced response.

First-Order System

General Transfer Function: $G(s) = \frac{K}{\tau s+1} = \frac{Ka}{s+a}$

Step response: $c(t) = K(1 - e^{-at}) = K(1 - e^{-t/\tau})$

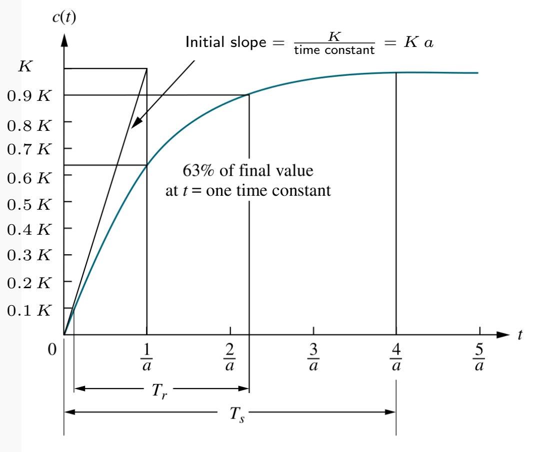
Time constant: $\tau = \frac{1}{a}$

Rise time T_r:

$$T_r = \frac{2.2}{a} = 2.2\tau$$

Settling time T_s (2%):

$$T_s = \frac{4}{a} = 4\tau$$

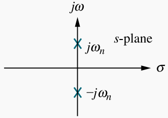
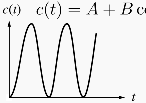
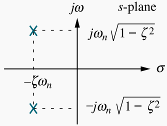
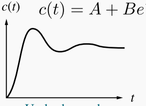
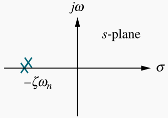
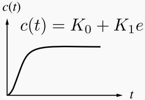
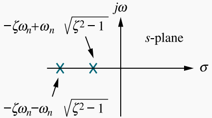
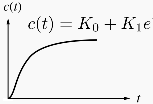


Second-Order System: $G(s) = \frac{K}{\frac{s^2}{\omega_n^2}+\frac{2\zeta}{\omega_n}s+1} = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

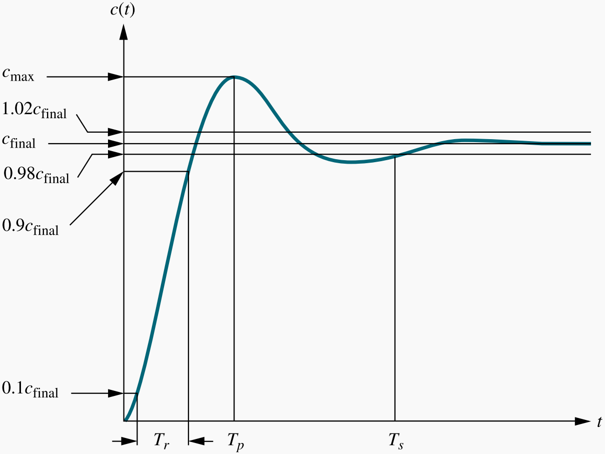
$$G = \frac{K}{\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1}$$

Roots:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

ζ	Poles	Position	Step response
0		Two imaginary at $\pm j\omega_1$	$c(t) = A + B\cos(\omega_1 t - \phi)$  Undamped
$0 < \zeta < 1$		Two complex at $-\sigma_d \pm j\omega_d$	$c(t) = A + Be^{-\sigma_d t}\cos(\omega_d t - \phi)$  Underdamped
$\zeta = 1$		Two real at $-\sigma_1$	$c(t) = K_0 + K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$  Critically damped
$\zeta > 1$		Two real at $-\sigma_1, -\sigma_2$	$c(t) = K_0 + K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$  Overdamped

2nd Order System: Specifications for underdamped response.



Second-order underdamped response specifications

Rise time: T_r

Peak time: T_p

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Percent overshoot: $\%OS$

$$\%OS = \frac{c_{\text{max}} - c_{\text{final}}}{c_{\text{final}}} \times 100$$

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100$$

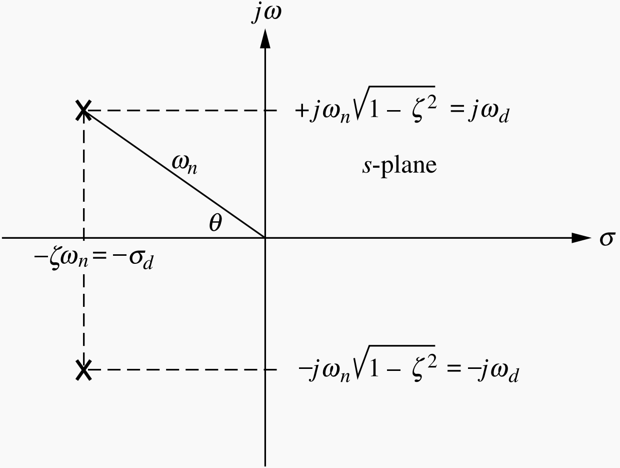
$$\zeta = \frac{-\ln(OS)}{\sqrt{\pi^2 + \left(\ln(OS)\right)^2}}$$

Settling time 2% :

$$T_s = \frac{4}{\zeta\omega_n}$$

Underdamped system: ω_n and ζ from pole plot

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



$\omega_n =$ radial distance from origin to the pole.

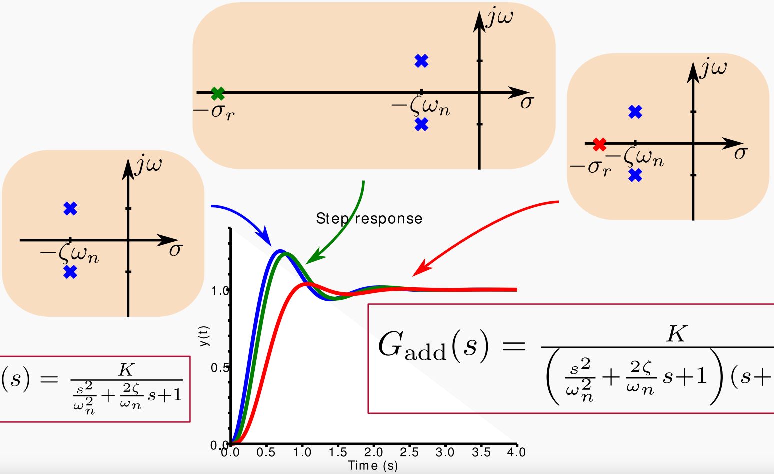
$\zeta = \cos(\theta)$

ω_d : Damped frequency of oscillation

$$T_p = \frac{\pi}{\omega_d}$$

$$T_s = \frac{4}{\sigma_d}$$

2nd Order Approximation: Dominant poles $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

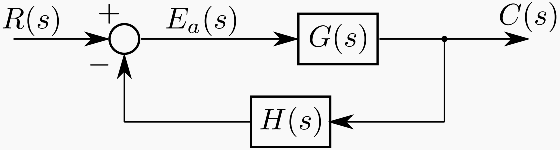


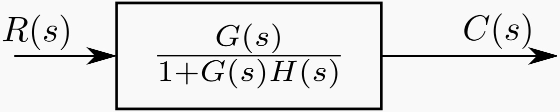
$$G(s) = \frac{K}{\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1}$$

$$G_{\text{add}}(s) = \frac{K}{\left(\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1\right)(s + \sigma_r)}$$

Approximation to a 2nd order
i.e. $G_{\text{add}}(s) \simeq G(s) \quad \Leftrightarrow \quad \sigma_r > 5\zeta\omega_n$

Feedback Form





Closed-loop transfer function

$$G_e(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

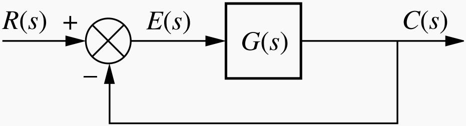
Steady-state error

Steady-State Error is the difference between the input and the output for a prescribed input as $t \to \infty$.

Steady-State Error for a system with input $R(s)$ and output $C(s)$

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0}(sE(s)) = \lim_{s\to 0}(s(R(s) - C(s)))$$

Steady-state error for a unity feedback system:



$$E(s) = R(s) - C(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)}$$

$$e_{ss} = e(\infty) = \lim_{s\to 0}(sE(s)) = \lim_{s\to 0}\left(\frac{sR(s)}{1+G(s)}\right)$$

Key equations: Static Error Constant

Position Constant:

$$K_p = \lim_{s \to 0} G(s)$$

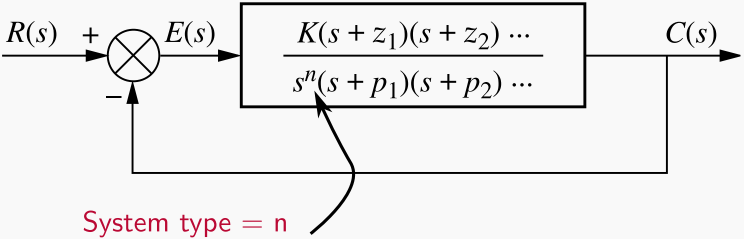
Velocity Constant

$$K_v = \lim_{s \to 0} sG(s)$$

Acceleration Constant:

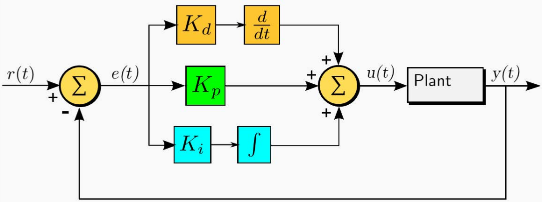
$$K_a = \lim_{s \to 0} s^2 G(s)$$

Key equations: Steady-State errors for Unity feedback systems



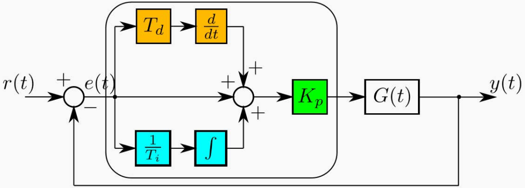
Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

PID



$$u(t) = \left(K_p e(t) + K_d \frac{de(t)}{dt} + K_i \int e(t)dt \right)$$

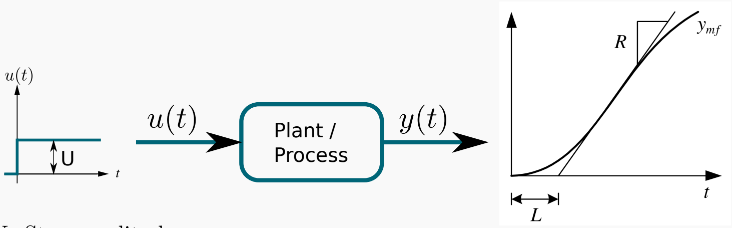
K_p: Proportional gain
K_i: Integral gain
K_d: Derivative gain



$$u(t) = K_p \left(e(t) + T_d \frac{de(t)}{dt} + \frac{1}{T_i} \int e(t)dt \right)$$

T_i: Integral time
T_d: Derivative time

Ziegler-Nichols Open-loop Tuning



U: Step amplitude

L: Dead time
R: Reaction rate (Slope)

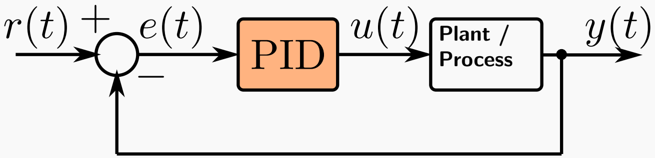
1. Using a step input of amplitude U, plot the step response (output).
2. Measure the dead-time L and reaction time R.
   - L is the time from the step time to the point of intersection between the 0.0-line and the steepest tangent.
   - R is the slope of the steepest tangent.
3. Calculate the controller parameter values according to Table, and use these parameter values in the PID controller.

Ziegler-Nichols Open-loop Tuning.
PID gains.

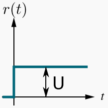
PID Type	$\mathbf{K_p}$	$\mathbf{T_i}(=K_p/K_i)$	$\mathbf{T_d}(=K_d/K_p)$
P	$\frac{1}{LR/U}$	∞	0
PI	$\frac{0.9}{LR/U}$	$3.3L$	0
PID	$\frac{1.2}{LR/U}$	$2L$	$0.5L$

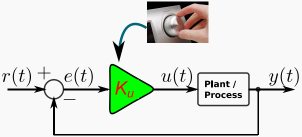
Control law: (output of the PID controller)

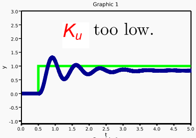
$$u(t) = K_p\left(e(t) + T_d\frac{de(t)}{dt} + \frac{1}{T_i}\int e(t)dt\right)$$

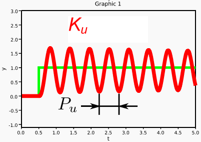


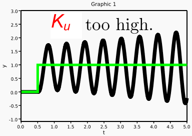
Ziegler-Nichols Closed-loop Tuning.
Experiment.











U: Step amplitude

K_u: Ultimate Gain
P_u: Ultimate Period

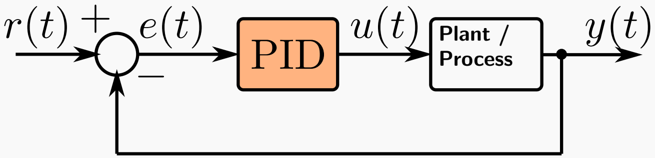
1. Using a step input of amplitude U.
2. Applying a Proportional control to the system.
3. Increasing the gain in the P controller until the output gives stable oscillations. This value is denoted the ultimate gain, K_u.
4. Measure the period P_u of the oscillation.
5. Calculate the controller parameter values according to Table, and use these parameter values in the PID controller.

Ziegler-Nichols Closed-loop Tuning.
PID gains.

PID Type	$\mathbf{K_p}$	$\mathbf{T_i} = K_p/K_i$	$\mathbf{T_d} = K_d/K_p$
P	$0.5K_u$	∞	0
PI	$0.45K_u$	$\frac{P_u}{1.2}$	0
PID	$0.6K_u$	$\frac{P_u}{2}$	$\frac{P_u}{8}$

Control law: (output of the PID controller)

$$u(t) = K_p\left(e(t) + T_d\frac{de(t)}{dt} + \frac{1}{T_i}\int e(t)dt\right)$$



Analytical Expression for Frequency Response

The frequency response of a system

$$G(j\omega) = G(s)|_{s \to j\omega}$$

The analytical expression of the frequency response of a system with transfer function $G(s)$ is:

$$M_G \angle \phi_G = G(j\omega)$$

- The magnitude frequency response is $M_G = |G(j\omega)|$
- The phase frequency response is $\phi_G =$ angle of $G(j\omega)$.

Bode Plots

Bode Plots: 2 separate logarithmic plots

- Magnitude plot: Amplitude in decibels (dB), where $\text{dB} = 20\log_{10}|G(j\omega)|$ vs. frequency $\log_{10}\omega$
- Phase plot: Phase angle in degrees, $\angle G(j\omega)$ vs. frequency $\log_{10}\omega$

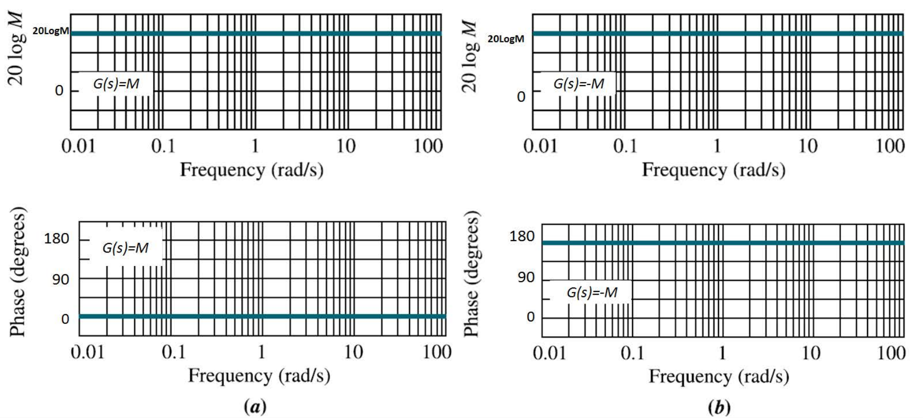
Common Asymptotic, $G(s) = M$ and $G(s) = -M$, $M > 0$

Gain $G(s) = M$

- Amplitude= $20 \log M$
- Phase=0^o

Gain $G(s) = -M$

- Amplitude= $20 \log M$
- Phase=180^o



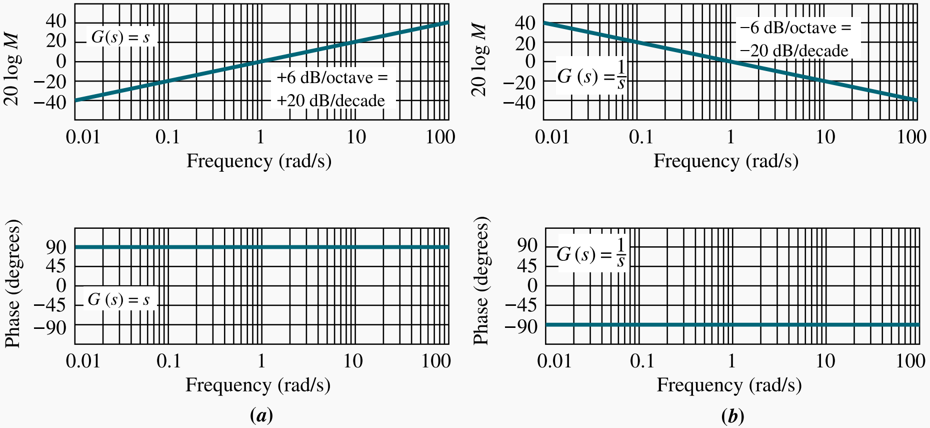
Common Asymptotes, $G(s) = s$ and $G(s) = \frac{1}{s}$

Differential $G(s) = s$

- Slope $= +20$ dB/decade (rise)
- Phase $= +90^o$

Integration $G(s) = \frac{1}{s}$

- Slope $= -20$ dB/decade (fall)
- Phase $= -90^o$



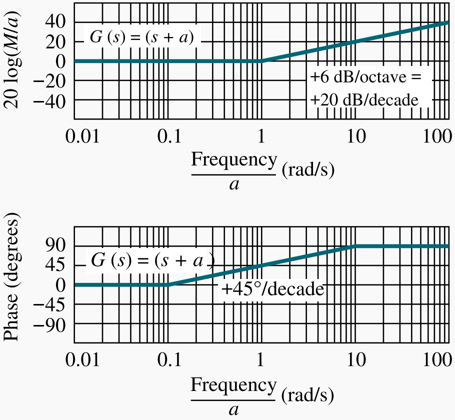
Common Asymptotes, 1st-order $G(s) = s + a$ and $G(s) = \frac{1}{s+a}$

1st-order zero $G(s) = s + a$

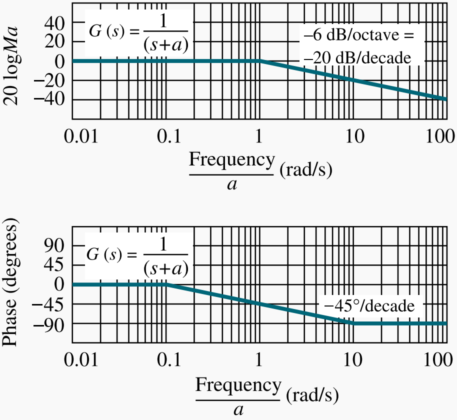
- Break frequency $= a$
- Magnitude Slope $= +20$ dB/decade
- Phase Slope $= +45^o$ /decade
- Phase shift 90^o

1st-order pole $G(s) = \frac{1}{s+a}$

- Break frequency $= a$
- Magnitude Slope $= -20$ dB/decade
- Phase Slope $= -45^o$ /decade
- Phase shift -90^o



(c)



(d)

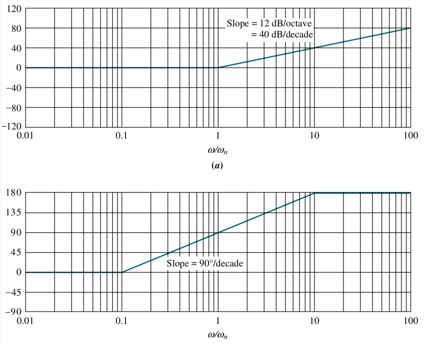
Common Asymptotes, 2nd-order approximation

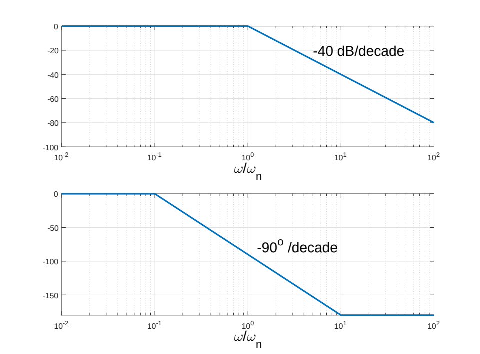
2nd-order zero $G(s) = \left(\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1\right)$

- Break frequency $= \omega_n$
- Magnitude Slope $= +40$ dB/decade
- Phase Slope $= +90^o$ /decade
- Phase shift 180^o

2nd-order pole $G(s) = \frac{1}{\left(\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1\right)}$

- Break frequency $= \omega_n$
- Magnitude Slope $= -40$ dB/decade
- Phase Slope $= -90^o$ /decade
- Phase shift -180^o

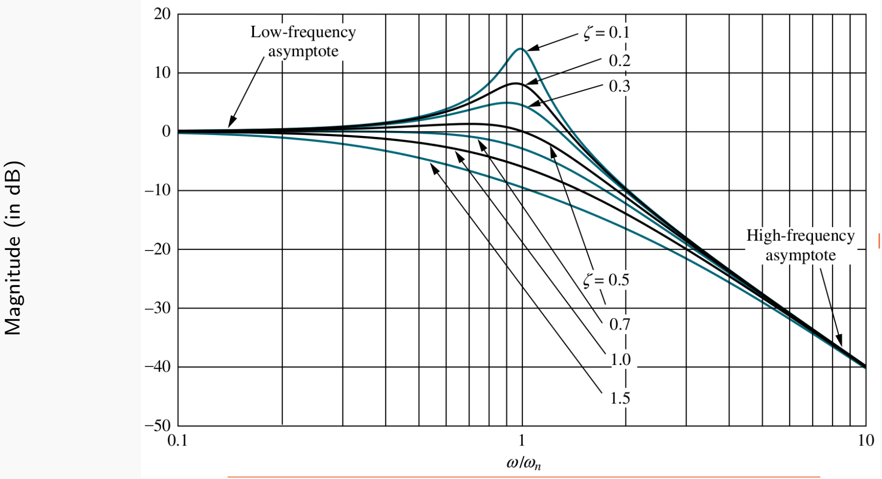




Second-Order Pole Bode plot - Magnitude G_{dB}

$$G(s) = \frac{1}{\left(\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1\right)} \Rightarrow G(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2} + j2\zeta\frac{\omega}{\omega_n}\right)}$$

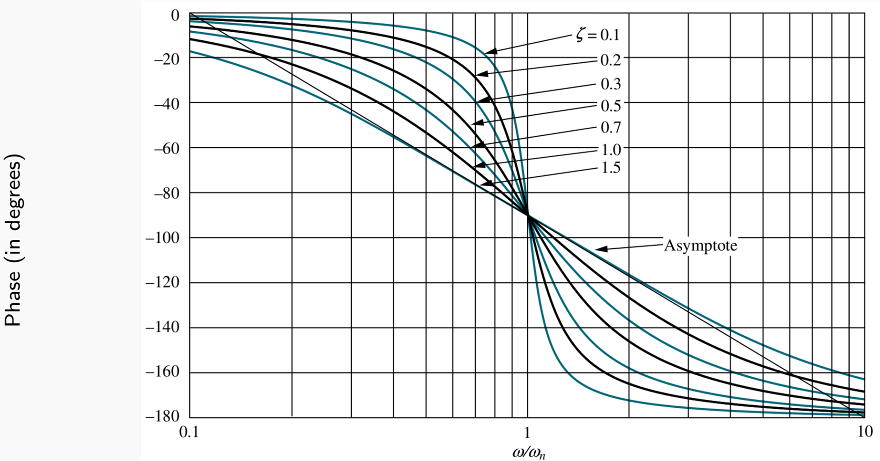
$$M = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$



Second-Order Pole Bode plot - Phase θ

$$G(s) = \frac{1}{\left(\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1\right)} \Rightarrow G(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2} + j2\zeta\frac{\omega}{\omega_n}\right)}$$

$$\theta = -\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)$$



System Identification

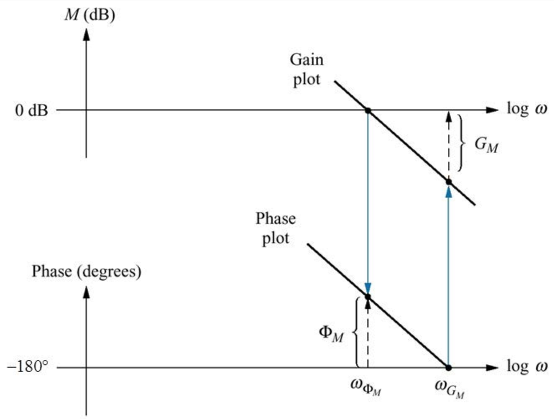
1. Plot amplitude and phase data on the Bode Plot
2. Fit straight-lines to amplitude data.
   - Slope must be 0 dB/decade, ± 20 dB/decade, ± 40 dB/decade, etc.
   - Straight-lines must pass through the data points at the lowest frequency and highest frequency.
3. Use relationships to determine the transfer function

System Identification

- Check asymptotic magnitude plot and phase plot in low frequency
  - if the initial magnitude plot with slope -20 dB/dec and the initial phase at -90°
    $\Rightarrow$ the transfer function has a $\frac{1}{s}$.
  - else if the initial magnitude plot with slope -40 dB/dec and the initial phase at -180°
    $\Rightarrow$ the transfer function has $\frac{1}{s^2}$.
  - else if the initial magnitude plot with slope +20 dB/dec and the initial phase at $+90^\circ$
    $\Rightarrow$ the transfer function has s.
  - else if ....
- Check the following slope in the magnitude plot and phase shift in phase plot, then find each break frequency a (Intersection of two lines), from the lowest to the highest,
  - If the magnitude slope change by +20 dB/dec and the phase shift is +90 degree,
    $\Rightarrow$ then the transfer function has $(s+a)$.
  - Elseif the magnitude slope change by -20 dB/dec and the phase shift is -90 degree,
    $\Rightarrow$ then the transfer function has $\frac{1}{(s+a)}$.
  - Elseif the magnitude slope change by +40 dB/dec and the phase shift is +180 degree,
    $\Rightarrow$ then the transfer function has $(s+a)^2$.
  - Elseif the magnitude slope change by -40 dB/dec and the phase shift is -180 degree,
    $\Rightarrow$ then the transfer function has $\frac{1}{(s+a)^2}$.
  - ....

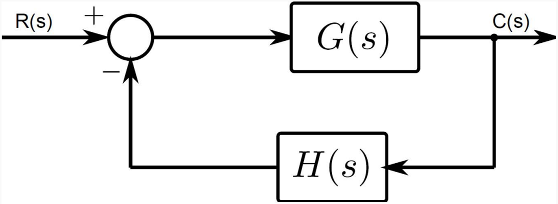
Gain and Phase Margins via Bode Plots

- Gain margin G_M: is the gain required to raise the magnitude curve to 0 dB.
    - Phase plot: find the frequency ω_{G_M}, where the phase angle is -180^o.
    - Magnitude plot: determine the gain margin G_M at this frequency.
- Phase margin Φ_M: is the difference between the phase value and -180^o
    - Magnitude plot: find the frequency ω_{Φ_M} where the gain is 0 dB.
    - Phase plot: determine the phase margin ϕ_M at this frequency.



Stability via Bode Plot

Determine the stability of the closed-loop system



Bode Plots of Open-loop Transfer Function $G(s)H(s)$

- The magnitude is less than 0 dB (unity gain) at the frequency where the phase is -180^o. $\Rightarrow$ The closed-loop system is Stable.
- Gain margin and phase margin are positive. $\Rightarrow$ The closed-loop system is Stable.

Key equations: Closed-loop Time Response and Open-loop Frequency response

Second order approximation:

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + (\ln(\%OS/100))^2}} \tag{1}$$

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 \tag{2}$$

$$\phi_M = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}} \tag{3}$$

$$\zeta = \frac{\sin(\phi_M)}{2\sqrt{\cos(\phi_M)}}, \quad 0 < \phi_M < 90^o \tag{4}$$

$$\omega_{BW} = \omega_n\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \tag{5}$$

$$\omega_{BW} = \frac{4}{T_s\zeta}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \tag{6}$$

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \tag{7}$$

$$\omega_{BW} = \omega_{OL|G_{dB}\in[-6dB,-7.5dB]} \text{ if } \phi_{OL} \in [-135^\circ, -225^\circ] \tag{8}$$

Key equations: Static Error Constants from Open-loop Frequency

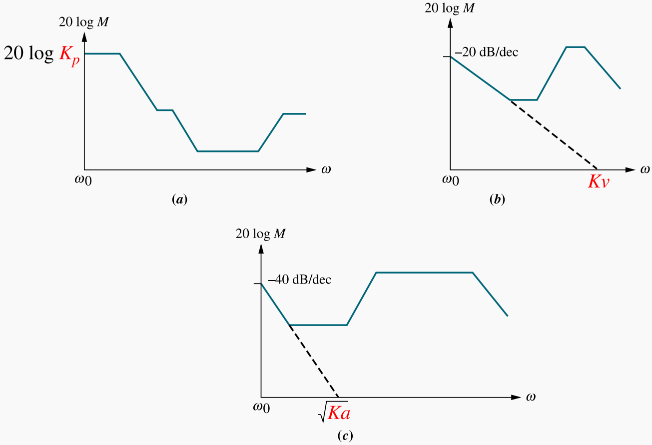
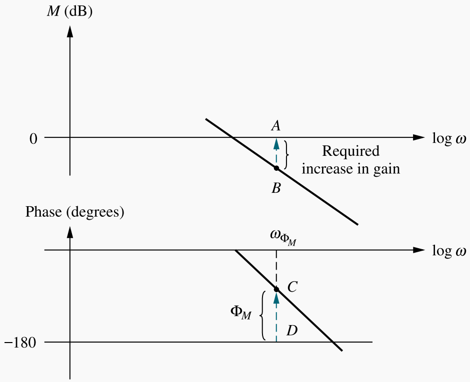


Figure: Bode log-magnitude plots showing the value of static error constants **a.** Type 0, **b.** Type 1, **c.** Type 2 $G(s) = K\frac{\prod_{i=1}^{k}(s+z_i)}{s^n \prod_{i=1}^{r}(s+p_i)}$

Position Constant: $K_p = \lim_{s\to 0} G(s)$.
Velocity Constant: $K_v = \lim_{s\to 0} sG(s)$.
Acceleration Constant: $K_a = \lim_{s\to 0} s^2 G(s)$

P-controller Design (Gain adjustment) Procedure: (to satisfy the required percent overshoot)

1. Draw the Bode plot for a convenient value of gain, for example $K = 1$.
2. Using second order approximation, determine the required phase margin ϕ_M from the percent overshoot.



3. Locate the frequency, ω_{ϕ_M} , on the Bode phase diagram that yields the required phase margin ϕ_M, CD
4. Adjust the gain to yield the desired phase margin, $20\log(K) = AB$.

P-controller Design Procedure via bode plots: **(to satisfy the required steady-state error)**

1. Using second order approximation, determine the required static error constant $K_{p,v,a}^{required}$ from the required steady-state error e_{ss}.
2. Read from bode plots, find the static error constant $K_{p,v,a}^{system}$ of current open-loop system.
3. Adjust the gain K to yield the desired steady-state error.

$$K = \frac{K_{p,v,a}^{required}}{K_{p,v,a}^{system}}$$

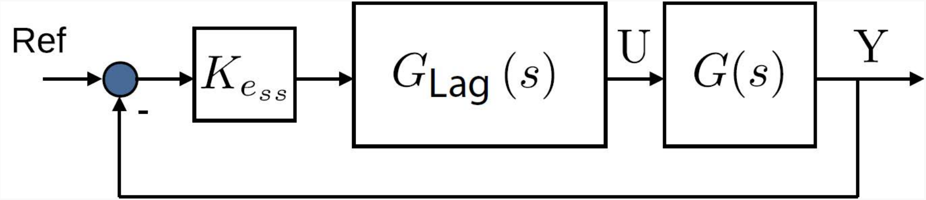
Lag Compensation

Goal: Increase the phase margin without affecting the steady-state error.

$$G_{Lag}(s) = \frac{b}{a}\left(\frac{s+a}{s+b}\right), \quad 0 < b < a$$

$$G_{C_{Lag}}(s) = K_{e_{ss}}\frac{b}{a}\left(\frac{s+a}{s+b}\right), \quad 0 < b < a$$

- Lag compensator consists of zero $-a$ and pole $-b$.
- $b < a$.



Lag Compensation

Goal: Reduce the steady-state error and improve the transient response (i.e. keeping the stability margins).

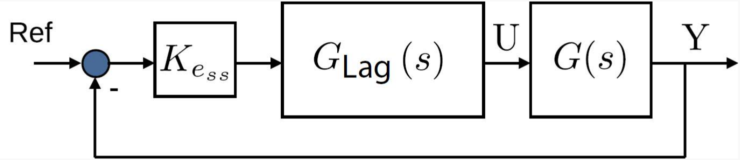
$$G_{Lag}(s) = \frac{b}{a}\left(\frac{s+a}{s+b}\right), \quad 0 < b < a$$

$$G_{\text{Lag}} = 10^{-\frac{G_{\text{adj}}}{20}}\left(\frac{s+0.1\omega_p}{s+0.1\omega_p \times 10^{-\frac{G_{\text{adj}}}{20}}}\right)$$

$$G_{c_{\text{Lag}}} = K_{ess} G_{\text{Lag}}$$

where:

$K_{e_{ss}}$	Gain adjustment in order to meet the required steady-state error.
ω_p	Frequency at which the phase margin would be the desired one plus 5° to 12° extra.
G_{adj}	Gain at $\omega = \omega_p$
a, b	$a = 0.1\omega_p$ and $b = 0.1\omega_p \times 10^{-\frac{G_{\text{adj}}}{20}}$



Lag Compensator-Design Procedure with Steady-State Error and Phase Margin Requirements

1. Set the gain K_{ess} to the value that satisfies the steady-state error specification. Then draw the Bode plot of the open-loop system with this gain K_{ess}.
2. Calculate the desired phase margin ϕ_M to meet the percent overshoot requirement.
3. Find the frequency ω_p where the phase margin is 5° to 12° greater than the desired phase margin. (for example, $\phi_M + 10$ degree)
4. Find the Gain G_{adj} at this frequency ω_p and insert the values into the following equations.

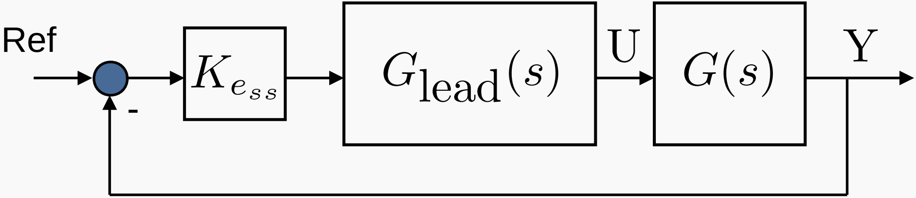
$$G_{c_{\text{Lag}}} = K_{e_{\text{ss}}} 10^{-\frac{G_{\text{adj}}}{20}} \left(\frac{s + 0.1\omega_p}{s + 0.1\omega_p \times 10^{-\frac{G_{\text{adj}}}{20}}} \right)$$

where:

$K_{e_{\text{ss}}}$	Gain adjustment in order to meet the required steady-state error.
ω_p	Frequency at which the phase margin would be the desired phase margin plus 5° to 12° extra.
G_{adj}	Gain in dB at $\omega = \omega_p$

Lead Compensation

Goal: Reduce percentage overshoot and realize a faster transient response.



$$G_{\text{Lead}}(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \qquad \beta < 1$$

$$G_{c_{\text{Lead}}}(s) = K_{ess}\frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \qquad \beta < 1$$

Lead Compensator: Key Equations and Structure

Second order approximation:

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} \tag{K.1}$$

$$\phi_M = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}} \tag{K.2}$$

$$\omega_{BW} = \frac{4}{T_s \zeta} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \tag{K.3}$$

$$\omega_{BW} = \frac{\pi}{T_p \sqrt{1-\zeta^2}} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \tag{K.4}$$

$$\omega_{BW} = \omega_{OL|G_{dB} \in [-6dB, -7.5dB]} \quad \text{if } \phi_{OL} \in [-135^\circ, -225^\circ] \tag{K.5}$$

Lead compensation:

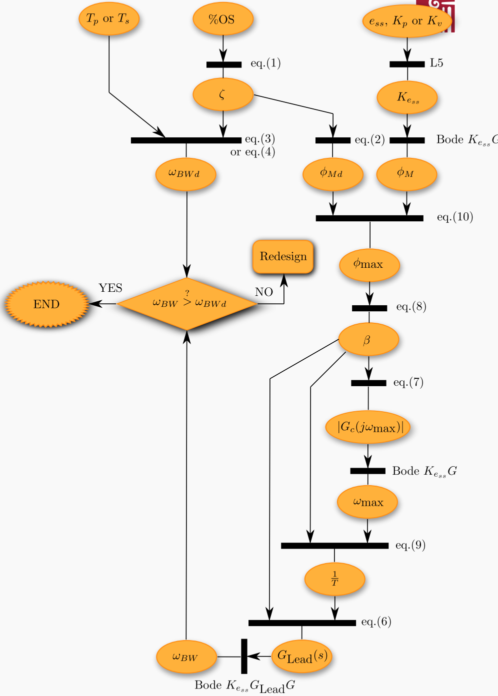
$$G_{\text{Lead}}(s) = \frac{1}{\beta} \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} \qquad \beta < 1 \tag{K.6}$$

$$|G_c(j\omega_{\text{max}})| = \frac{1}{\sqrt{\beta}} \tag{K.7}$$

$$\beta = \left(\frac{1 - \sin \phi_{\text{max}}}{1 + \sin \phi_{\text{max}}} \right) \tag{K.8}$$

$$\omega_{\text{max}} = \frac{1}{T\sqrt{\beta}} \tag{K.9}$$

$$\phi_{\text{max}} = \phi_{Md} - \phi_M + 10^\circ \tag{K.10}$$



Lead Compensator -Design Procedure with Steady-State Error and Phase Margin Requirements

1. Set the gain, K_{ess}, to satisfy the steady-state error requirement.
2. Plot the Bode magnitude and phase plots for this value of gain K_{ess} and determine the uncompensated system's phase margin ϕ_M.
3. Calculate the desired phase margin ϕ_{Md} to meet the percent overshoot requirement. Then evaluate the additional phase contribution required from the compensator.
   $\phi_{max} = \phi_{Md} - \phi_M + 10^o$ (add $[5^o \quad 12^o]$)
4. Determine the value of β (see Eq. (K.8)) from the lead compensators required phase contribution ϕ_{max}.
5. Determine the compensators magnitude $|G_c(j\omega_{\text{max}})|$ at the peak of the phase curve (see Eq. (K.7))
6. Determine ω_{max} by finding where the uncompensated systems magnitude curve is the negative of $|G_c(j\omega_{\text{max}})|$.
7. Find T from Eq. (K.9)
8. Lead compensator: $G_{\text{Lead}}(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$.
9. Plot Bode for compensated system. Check the bandwidth to be sure the speed requirement has been met. Redesign if not satisfied. (Calculate the bandwidth to meet the settling time, peak time, or rise time requirement.)

Alternative Design: Lead Compensator with Steady-State Error and Cross-Frequency Requirements

1. Set the gain, K_{ess}, to satisfy the steady-state error requirement.
2. The maximum reachable phase margin is selected by choosing:

$$\omega_{\mathsf{max}} = \omega_c$$

3. Reading from bode plots of $K_{ess}G(S)$, find the gain $|G_c(j\omega_{\mathsf{max}})|$ at frequency ω_{max}.
4. Calculate the value of β from Eq. (K.7).
5. Calculate the value of T from Eq. (K.9).
6. Lead compensator: $G_{\mathsf{Lead}}(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$.